

# On classification of non-unital amenable simple $C^*$ -algebras, II

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## Abstract

We present a classification theorem for amenable simple stably projectionless  $C^*$ -algebras with generalized tracial rank one whose  $K_0$  vanish on traces which satisfy the Universal Coefficient Theorem. One of them is denoted by  $\mathcal{Z}_0$  which has a unique tracial state and  $K_0(\mathcal{Z}_0) = \mathbb{Z}$  and  $K_1(\mathcal{Z}_0) = \{0\}$ . Let  $A$  and  $B$  be two separable simple  $C^*$ -algebras satisfying the UCT and have finite nuclear dimension. We show that  $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$  if and only if  $\text{Ell}(B \otimes \mathcal{Z}_0) = \text{Ell}(A \otimes \mathcal{Z}_0)$ . A class of simple separable  $C^*$ -algebras which are approximately sub-homogeneous whose spectra having bounded dimension is shown to exhaust all possible Elliott invariant for  $C^*$ -algebras of the form  $A \otimes \mathcal{Z}_0$ , where  $A$  is any finite separable simple amenable  $C^*$ -algebras. Suppose that  $A$  and  $B$  are two finite separable simple  $C^*$ -algebras with finite nuclear dimension satisfying the UCT such that both  $K_0(A)$  and  $K_0(B)$  are torsion (but arbitrary  $K_1$ ). One consequence of the main results in this situation is that  $A \cong B$  if and only if  $A$  and  $B$  have the isomorphic Elliott invariant.

## 1 Introduction

Recently some sweeping progresses have been made in the Elliott program ([10]), the program of classification of separable amenable  $C^*$ -algebras by the Elliott invariant (a  $K$ -theoretical set of invariant) (see [18], [52] and [14]). These are the results of decades of work by many mathematicians (see also [18], [52] and [14] for the historical discussion there). These progresses could be summarized briefly as the following: Two unital finite separable simple  $C^*$ -algebras  $A$  and  $B$  with finite nuclear dimension which satisfy the UCT are isomorphic if and only if their Elliott invariant  $\text{Ell}(A)$  and  $\text{Ell}(B)$  are isomorphic. Moreover, all weakly unperforated Elliott invariant can be achieved by a finite separable simple  $C^*$ -algebras in UCT class with finite nuclear dimension (In fact these can be constructed as so-called ASH-algebras—see [18]). Combining with the previous classification of purely infinite simple  $C^*$ -algebras, results of Kirchberg and Phillips ([40] and [24]), now all unital separable simple  $C^*$ -algebras in the UCT class with finite nuclear dimension are classified by the Elliott invariant.

This research studies the non-unital cases.

Suppose that  $A$  is a separable simple  $C^*$ -algebra. In the case that  $K_0(A)_+ \neq \{0\}$ , then  $A \otimes \mathcal{K}$  has a non-zero projection, say  $p$ . Then  $p(A \otimes \mathcal{K})p$  is unital. Therefore if  $A$  is in the UCT class and has finite nuclear dimension, then  $p(A \otimes \mathcal{K})p$  falls into the class of  $C^*$ -algebras which has been classified. Therefore isomorphism theorem for these  $C^*$ -algebras is an immediate consequence of that in [18] (see section 8.4 of [35]) using the stable isomorphism theorem of [3].

Therefore this paper considers the case that  $K_0(A)_+ = \{0\}$ . Simple  $C^*$ -algebras with  $K_0(A)_+ = \{0\}$  are stably projectionless in the sense that not only  $A$  has no non-zero projections but  $M_n(A)$  also has no non-zero projections for every integer  $n \geq 1$ . However, as one may see in this paper,  $K_0(A)$  could still exhaust any countable abelian groups as well as any possible  $K_0(A)$ . In particular, the results in [18] cannot be applied in the stably projectionless case. It is entirely new situation.

In the first part of this research, we introduce a class of stably projectionless simple  $C^*$ -algebras  $\mathcal{D}$  (see 3.12 below). We also introduced the notion of generalized tracial rank one

for stably projectionless simple  $C^*$ -algebras. These are separable stably projectionless simple  $C^*$ -algebras which are stably isomorphic to  $C^*$ -algebras in  $\mathcal{D}$  (see 3.12 below). If  $A$  is stably isomorphic to one in  $\mathcal{D}$ , we will write  $gTR(A) \leq 1$ . Some study of the structure of these  $C^*$ -algebras were also presented in the first part of this research. For example, among other things, we show that  $C^*$ -algebras have stable rank one. Let  $A$  and  $B$  be two stably projectionless simple amenable  $C^*$ -algebras satisfy the UCT. Suppose that  $K_0(A) = K_1(A) = K_0(B) = K_1(B) = \{0\}$ . In the first part of this research, we show that  $A \cong B$  if and only if  $\text{Ell}(A) \cong \text{Ell}(B)$  (This result was also independently obtained in [13]). In this case the Elliott invariant is reduced to  $\text{Ell}(A) = (\tilde{T}(A), \Sigma_A)$  (see 2.9 below). Combining the above mentioned result, this also gives a classification for separable stably finite projectionless simple  $C^*$ -algebras with finite nuclear dimension in the UCT class with trivial  $K_i$ -theory.

In this part of the research, we study the general case that  $K$ -theory of  $C^*$ -algebras are non-trivial. We give the following theorem:

**Theorem 1.1.** (see 12.2) *Let  $A$  and  $B$  be two separable simple amenable  $C^*$ -algebras which satisfy the UCT. Suppose that  $gTR(A) \leq 1$  and  $gTR(B) \leq 1$  and  $K_0(A) = \ker \rho_A$  and  $K_0(B) = \ker \rho_B$ . Then  $A \cong B$  if and only if*

$$\text{Ell}(A) \cong \text{Ell}(B). \quad (\text{e } 1.1)$$

Among all stably projectionless separable simple  $C^*$ -algebras, one particularly interesting one is  $W$ , a separable  $C^*$ -algebra with only one tracial state such that  $K_0(W) = K_1(W) = \{0\}$ .  $W$  is also an inductive limit of sub-homogeneous  $C^*$ -algebras (see [41]). It was shown in the part I ([17]) of this research that if  $A$  is a separable simple  $C^*$ -algebra in the UCT class, with finite nuclear dimension, with a unique tracial state and zero  $K_i(A)$ , then  $A \cong W$ .

In this part of the research, another stably projectionless simple  $C^*$ -algebra  $\mathcal{Z}_0$  with a unique tracial state plays a prominent role. This  $C^*$ -algebra has the property that  $K_0(\mathcal{Z}_0) = \mathbb{Z}$  and  $K_1(\mathcal{Z}_0) = \{0\}$ . As abelian groups,  $K_i(\mathcal{Z}_0) = K_i(\mathbb{C})$ ,  $i = 0, 1$ . Therefore, by Künneth Formula, for any separable  $C^*$ -algebra  $A$ ,  $K_i(A \otimes \mathcal{Z}_0) = K_i(A)$ , as abelian group,  $i = 0, 1$ . Moreover, if the tracial state space of  $A$  is not empty, then  $T(A \otimes \mathcal{Z}_0) = T(A)$ , since  $\mathcal{Z}_0$  has only one tracial state. As consequence of our main results,  $\mathcal{Z}_0 \otimes \mathcal{Z}_0 \cong \mathcal{Z}_0$ . Therefore we are particularly interested in the  $\mathcal{Z}_0$ -stable  $C^*$ -algebras, i.e., those  $C^*$ -algebras with the property that  $A \otimes \mathcal{Z}_0 \cong A$ .

We prove the following theorem:

**Theorem 1.2.** (see 14.4) *Let  $A$  and  $B$  be two separable simple  $C^*$ -algebras with finite nuclear dimension which satisfies the UCT. Then  $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$  if and only if*

$$\text{Ell}(A \otimes \mathcal{Z}_0) \cong \text{Ell}(B \otimes \mathcal{Z}_0). \quad (\text{e } 1.2)$$

When  $A$  and  $B$  are infinite, then both  $A \otimes \mathcal{Z}_0$  and  $B \otimes \mathcal{Z}_0$  are purely infinite simple. This case is covered by Kirchberg-Phillips classification theorem (see [24] and [40]).

We also present models for  $C^*$ -algebras stably isomorphic to  $C^*$ -algebras in  $\mathcal{D}$ . These model  $C^*$ -algebras are locally approximated by sub-homogeneous  $C^*$ -algebras whose spectra have dimension no more than 3. We show that these  $C^*$ -algebras exhaust all possible Elliott invariant for separable  $\mathcal{Z}_0$ -stable  $C^*$ -algebras as stated as follows (see also 6.12 below):

**Theorem 1.3.** (see 7.4) *Let  $A$  be a finite separable simple amenable  $C^*$ -algebra. Then there exists a stably projectionless simple  $C^*$ -algebra  $B$  which is locally approximated by sub-homogeneous  $C^*$ -algebras and which is stably isomorphic to a  $C^*$ -algebra in  $\mathcal{D}$  such that*

$$\text{Ell}(A \otimes \mathcal{Z}_0) = \text{Ell}(B). \quad (\text{e } 1.3)$$

Finally, we show that the condition that  $A$  and  $B$  has generalized tracial rank at most one in Theorem 1.1 can be replaced by finite nuclear dimension. In fact, we have the following:

**Theorem 1.4.** *Let  $A$  and  $B$  be two finite separable simple  $C^*$ -algebras with finite nuclear dimension which satisfy the UCT. Suppose that  $K_0(A)$  and  $K_0(B)$  are torsion. Then  $A \cong B$  if and only if  $\text{Ell}(A) \cong \text{Ell}(B)$ .*

The paper also includes an appendix which shows every separable and amenable  $C^*$ -algebra in  $\mathcal{D}$  is  $\mathcal{Z}$ -stable.

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## 2 Preliminaries

**Definition 2.1.** Let  $A$  be a unital separable amenable  $C^*$ -algebra and let  $x \in A$ . Suppose that  $\|xx^* - 1\| < 1$  and  $\|x^*x - 1\| < 1$ . Then  $x|x|^{-1}$  is a unitary. Let us use  $[x]$  to denote  $x|x|^{-1}$ .

Denote by  $U(A)$  the unitary group of  $A$  and denote by  $U_0(A)$  the normal subgroup of  $U(A)$  consisting of those unitaries which are path connected with  $1_A$ . Denote by  $CU(A)$  the closure of the commutator subgroup of  $U(A)$ .

If  $u \in A$  is a unitary, then  $\bar{u}$  is the image of  $u$  in  $U(A)/CU(A)$ , and if  $\mathcal{U} \subset U(A)$  is a subset, then  $\bar{\mathcal{U}} = \{\bar{u} : u \in \mathcal{U}\}$ .

**Definition 2.2.** Let  $A$  be a  $C^*$ -algebra. Denote by  $A^1$  the unit ball of  $A$ .

Let  $B$  be another  $C^*$ -algebra and let  $\varphi : A \rightarrow B$  be a positive linear map. Suppose that  $r \geq 1$  be an integer. This map induces a positive linear map  $\varphi \otimes \text{id}_{M_r} : A \otimes M_r \rightarrow B \otimes M_r$ . Throughout this paper, we will use notation  $\varphi$  instead of  $\varphi \otimes \text{id}_{M_r}$  whenever it is convenient.

Let  $A$  be a non-unital  $C^*$ -algebra and let  $\varphi : A \rightarrow B$  (for some  $C^*$ -algebra  $B$ ) be a linear map. Throughout the paper, we will continue to use  $\varphi$  for the extension from  $\tilde{A}$  to  $\tilde{B}$ , whenever it is convenient.

**Definition 2.3.** Let  $A$  be a  $C^*$ -algebra. Denote by  $T(A)$  the tracial state of  $A$ . Let  $\text{Aff}(T(A))$  be the space of all real valued affine continuous functions on  $T(A)$  which could be an empty set. Let  $\tilde{T}(A)$  be the cone of densely defined, positive lower semi-continuous traces on  $A$  equipped with the topology of point-wise convergence on elements of the Pedersen ideal  $P(A)$  of  $A$ . Let  $B$  be another  $C^*$ -algebra with  $T(B) \neq \emptyset$  and let  $\varphi : A \rightarrow B$  be a homomorphism. We will use then  $\varphi_T : T(B) \rightarrow T(A)$  for the induced continuous affine map.

Let  $r \geq 1$  be an integer and  $\tau \in \tilde{T}(A)$ . We will continue to use  $\tau$  on  $A \otimes M_r$  for  $\tau \otimes \text{Tr}$ , where  $\text{Tr}$  is the standard trace on  $M_r$ . Let

$$\text{Aff}(\tilde{T}(A))_+ = \{f : C(\tilde{T}(A), \mathbb{R})_+ : f \text{ linear}, f(\tau) \geq 0\}, \quad (\text{e2.1})$$

$$\text{Aff}_+(\tilde{T}(A)) = \{f : C(\tilde{T}(A), \mathbb{R})_+ : f \text{ linear}, f(\tau) > 0 \text{ for } \tau \neq 0\}, \quad (\text{e2.2})$$

$$\text{LAff}_f(\tilde{T}(A))_+ = \{f : \tilde{T}(A) \rightarrow [0, \infty) : \exists \{f_n\}, f_n \nearrow f, f_n \in \text{Aff}(\tilde{T}(A))_+\}, \quad (\text{e2.3})$$

$$\text{LAff}_{f,+}(\tilde{T}(A)) = \{f : \tilde{T}(A) \rightarrow [0, \infty) : \exists \{f_n\}, f_n \nearrow f, f_n \in \text{Aff}_+(\tilde{T}(A))\}, \quad (\text{e2.4})$$

$$\text{LAff}(\tilde{T}(A))_+ = \{f : \tilde{T}(A) \rightarrow [0, \infty) : \exists \{f_n\}, f_n \nearrow f, f_n \in \text{Aff}(\tilde{T}(A))_+\}, \quad (\text{e2.5})$$

$$\text{LAff}_+(\tilde{T}(A)) = \{f : \tilde{T}(A) \rightarrow [0, \infty) : \exists \{f_n\}, f_n \nearrow f, f_n \in \text{Aff}_+(\tilde{T}(A))\} \text{ and } \quad (\text{e2.6})$$

$$\text{LAff}^\sim(\tilde{T}(A)) = \{f_1 - f_2 : f_1 \in \text{LAff}_+(\tilde{T}(A)) \text{ and } f_2 \in \text{Aff}_+(\tilde{T}(A))\}. \quad (\text{e2.7})$$

Moreover,  $\text{LAff}_{b+}(T(A))$  is the subset of those bounded functions in  $\text{LAff}_{f+}(T(A))$ .

**Definition 2.4.** Let  $A$  be a  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Let  $\pi_A : \tilde{A} \rightarrow \mathbb{C}$  be the quotient map and  $s : \mathbb{C} \rightarrow \tilde{A}$  be the homomorphism such that  $\pi \circ s = \text{id}_{\mathbb{C}}$ . Recall that we also use  $\pi_A$  for the induced homomorphism  $\pi_A \otimes \text{id}_{M_r} : M_r(\tilde{A}) \rightarrow M_r$  and use  $s$  for the induced homomorphism  $s \otimes \text{id}_{M_r} : M_r \rightarrow M_r(\tilde{A})$  for all integer  $r \geq 1$ . Let  $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$  be the order preserving homomorphism defined by  $\rho([p] - [s \circ \pi_A(p)])(\tau) = \tau(p - s \circ \pi_A(p))$  for any projections in  $M_r(\tilde{A})$  for all integer  $r \geq 1$ .

Suppose that  $A$  is non-unital and separable and  $\tilde{T}(A) \neq \emptyset$ . Suppose that there exists  $a \in P(A)_+$  which is full. Let  $A_a = \overline{aAa}$ . Then  $T(A_a) \neq \emptyset$ . We define

$$\ker \rho_A = \{x \in K_0(A_a) : \rho_A(x) = 0\} \quad (\text{e 2.8})$$

Here we also identify  $K_0(A_a)$  with  $K_0(A)$  using the Brown's stable isomorphism theorem ([3]).

Suppose that  $A$  is unital and has stable rank one. Then we have (by [49] and [19]) the following splitting short exact sequence (we will fix one such  $J_c$ )

$$0 \longrightarrow \text{Aff}(T(A))/\overline{\rho_A(K_0(A))} \longrightarrow U(A)/CU(A) \rightleftarrows_{J_c} K_1(A) \longrightarrow 0. \quad (\text{e 2.9})$$

If  $u \in U_0(A)$  and  $\{u(t) : t \in [0, 1]\}$  is a piece-wise smooth and continuous path of unitaries in  $A$  such that  $u(0) = u$  and  $u(1) = 1$ . Then, for each  $\tau \in T(A)$ ,

$$D_A(u)(\tau) = \frac{1}{2\pi i} \int_0^1 \tau\left(\frac{du(t)}{dt} u(t)^*\right) dt \quad (\text{e 2.10})$$

modulo  $\overline{\rho_A(K_0(A))}$  induces (independent of the path) an isomorphism (denote by  $\bar{D}_A$ ) from  $U_0(A)/CU(A)$  onto  $\text{Aff}(T(A))/\overline{\rho_A(K_0(A))}$  as mentioned above (see also 2.15 of [18]).

Now suppose that  $A$  is a non-unital separable  $C^*$ -algebra and  $P(A) = A$  with  $T(A) \neq \emptyset$ . Suppose that  $\ker \rho_A = K_0(A)$ . Then

$$\text{Aff}(T(\tilde{A}))/\overline{\rho_A(K_0(\tilde{A}))} = \text{Aff}(T(\tilde{A}))/\mathbb{Z}. \quad (\text{e 2.11})$$

**Definition 2.5.** Let  $A$  be a non-unital  $C^*$ -algebra. We say  $A$  has almost stable rank one (see [43] and [17]) if, for each  $n$ , the invertible elements in  $M_n(\tilde{A})$  is dense in  $M_n(A)$ .

**Definition 2.6.** Let  $A$  be a unital separable amenable  $C^*$ -algebra. For any finite subset  $\mathcal{U} \subset U(A)$ , there exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset A$  satisfying the following: If  $B$  is another unital  $C^*$ -algebra and if  $L : A \rightarrow B$  is an  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map, then  $\overline{[L(u)]}$  is a well defined element in  $U(B)/CU(B)$  for all  $u \in \mathcal{U}$ . We will write  $L^\dagger(\bar{u}) = \overline{[L(u)]}$ . Let  $G(\mathcal{U})$  be the subgroup generated by  $\mathcal{U}$ . We may assume that  $L^\dagger$  is a well-defined homomorphism on  $G(\mathcal{U})$  so that  $L^\dagger(u) = \overline{[L(u)]}$  for all  $u \in \mathcal{U}$ . In what follows, whenever we write  $L^\dagger$ , we mean that  $\varepsilon$  is small enough and  $\mathcal{F}$  is large enough so that  $L^\dagger$  is well defined (see Appendix in [33]). Moreover, for an integer  $k \geq 1$ , we will also use  $L^\dagger$  for the map on  $U(M_k(A))/CU(M_k(A))$  induced by  $L \otimes \text{id}_{M_k}$ . In particular, when  $L$  is a unital homomorphism, the map  $L^\dagger$  is well defined on  $U(A)/CU(A)$ .

**Definition 2.7.** Let  $1 > \varepsilon > 0$ . Define

$$f_\varepsilon(t) = \begin{cases} f_\varepsilon(t) = 0, & \text{if } t \in [0, \varepsilon/2]; \\ f_\varepsilon(t) = \frac{t - \varepsilon/2}{\varepsilon/2}, & \text{if } t \in (\varepsilon/2, \varepsilon]; \\ f_\varepsilon(t) = 1 & \text{if } t \in (\varepsilon, \infty). \end{cases} \quad (\text{e 2.12})$$

**Definition 2.8.** Let  $A$  be a  $C^*$ -algebra and let  $a \in A_+$ . Suppose that  $\tilde{T}(A) \neq \emptyset$ . Recall that

$$d_\tau(a) = \lim_{\varepsilon \rightarrow 0} \tau(f_\varepsilon(a))$$

with possible infinite value. Note that  $f_\varepsilon(a) \in P(A)_+$ . Therefore  $\tau \mapsto d_\tau(a)$  is a lower semi-continuous affine function on  $\tilde{T}(A)$  (to  $[0, \infty]$ ). Suppose that  $A$  is non-unital. Let  $a \in A_+$  be a strictly positive element. Define

$$\Sigma_A(\tau) = d_\tau(a) \text{ for all } \tau \in \tilde{T}(A).$$

It is standard and routine to check that  $\Sigma_A$  is independent of the choice of  $a$ . The lower semi-continuous affine function  $\Sigma_A$  is called the scale function of  $A$ . (see 2.3 of [17]).

**Definition 2.9.** Let  $C_1$  and  $C_2$  be two cones. A cone map  $\gamma : C_1 \rightarrow C_2$  is a map such that  $\gamma(0) = 0$ ,  $\gamma(rc) = r\gamma(c)$  for all  $r \in \mathbb{R}_+$ .

Let  $A$  be a stably projectionless simple  $C^*$ -algebra such that  $K_0(A) = \ker \rho_A$ . Then the Elliott invariant is defined as follows:

$$\text{Ell}(A) = (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A).$$

Suppose that  $B$  is another stably projectionless simple  $C^*$ -algebra such that  $K_0(B) = \ker \rho_B$ . Then we write

$$\text{Ell}(A) \cong \text{Ell}(B),$$

if there are group isomorphisms  $\kappa_i : K_i(A) \rightarrow K_i(B)$ ,  $i = 0, 1$ , a cone homeomorphism  $\kappa_T : \tilde{T}(A) \rightarrow \tilde{T}(B)$ , i.e.,  $\kappa_T$  is 1-1 and onto,  $\kappa_T$  and  $\kappa_T^{-1}$  are both cone maps which are continuous (regarding weak\*-topology given by elements in  $P(A)$ ), and  $\Sigma_A(\tau) = \Sigma_B(\kappa_T(\tau))$  for all  $\tau \in \tilde{T}(A)$ . In case that  $A$  has continuous scale, then one can simplify

$$\text{Ell}(A) = (K_0(A), K_1(A), T(A)).$$

**Definition 2.10.** Let  $A$  and  $B$  be two  $C^*$ -algebras with  $T(A) \neq \emptyset$  and  $T(B) \neq \emptyset$  and both have stable rank one. Let  $\kappa \in KL(A, B)$ ,  $\kappa_T : T(B) \rightarrow T(A)$  be an affine continuous map and  $\kappa_u : U(\tilde{A})/CU(\tilde{A}) \rightarrow U(\tilde{B})/CU(\tilde{B})$ . We say  $(\kappa, \kappa_T, \kappa_u)$  are compatible, if  $\rho_B(\kappa(x))(t) = \rho_A(x)(\kappa_T(t))$  for all  $x \in K_0(A)$  and  $t \in T(B)$ ,  $\kappa(\pi_u(\bar{w})) = \kappa_u(\bar{w})$  for all  $\bar{w} \in U(A)/CU(A)$  and  $D_{\tilde{B}}(z)(t) = D_{\tilde{A}}(w)(\kappa_T(t))$  for all  $t \in T(B)$ , where  $w \in U_0(A)$ ,  $z \in U_0(B)$  such that  $\bar{z} = \kappa_u(\bar{w})$  for all  $w \in U_0(A)$ .

**Definition 2.11.** Let  $A$  and  $B$  be two separable  $C^*$ -algebras and let  $\varphi_n : A \rightarrow B$  be a sequence of linear maps. We say that  $\{\varphi_n\}$  is approximately multiplicative, if

$$\lim_{n \rightarrow \infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \text{ for all } a, b \in A. \quad (\text{e 2.13})$$

**Definition 2.12.** Throughout this paper,  $Q$  will be the universal UHF-algebra with  $K_0(Q) = \mathbb{Q}$ ,  $[1_Q] = 1$ .

**Definition 2.13.** Let  $\mathcal{B}$  be a class of  $C^*$ -algebras and let  $A$  be a separable  $C^*$ -algebra. We say  $A$  is *locally approximated by  $C^*$ -algebras in  $\mathcal{B}$* , if, for  $\varepsilon > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exists a  $C^*$ -algebra  $B \in \mathcal{B}$  such that  $\text{dist}(a, B) < \varepsilon$  for all  $a \in \mathcal{F}$ .

**Definition 2.14.** Let  $A$  be a  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Suppose that  $A$  has a strictly positive element  $e_A \in P(A)_+$  with  $\|e_A\| = 1$ . Then  $0 \notin \overline{T(A)}^w$ , the weak\*-closure of  $T(A)$  in  $\tilde{T}(A)$  (see section 8 of [17]).

$$\lambda_s(A) = \inf\{d_\tau(e_A) : \tau \in A\}.$$

Let  $A$  be a  $C^*$ -algebra with  $T(A) \neq \{0\}$  such that  $0 \notin \overline{T(A)}^w$ . There is an affine map  $r_{\text{aff}} : A_{s.a.} \rightarrow \text{Aff}(\overline{T(A)}^w)$  defined by

$$r_{\text{aff}}(a)(\tau) = \hat{a}(\tau) = \tau(a) \text{ for all } \tau \in \overline{T(A)}^w$$

and for all  $a \in A_{s.a.}$ . Denote by  $A_{s.a.}^q$  the space  $r_{\text{aff}}(A_{s.a.})$  and  $A_+^q = r_{\text{aff}}(A_+)$ .

**Definition 2.15.** (see 2.5 of [25]) Let  $A$  be a  $\sigma$ -unital, nonunital, non-elementary, simple  $C^*$ -algebra and  $\{e_n\}$  be an approximate identity such that  $e_{n+1}e_n = e_n$  for all  $n$ . We say  $A$  has continuous scale if, for any  $a \in A_+ \setminus \{0\}$ , there exists  $n_0 \geq 1$  such that  $e_m - e_n \lesssim a$  for all  $m \geq n \geq n_0$ .

### 3 Non-commutative 1-dimensional complices, revisited

**Definition 3.1** (See [16] and [11]). Let  $F_1$  and  $F_2$  be two finite dimensional  $C^*$ -algebras. Suppose that there are two (not necessary unital) homomorphisms  $\varphi_0, \varphi_1 : F_1 \rightarrow F_2$ . Denote the mapping torus  $M_{\varphi_1, \varphi_2}$  by

$$A = A(F_1, F_2, \varphi_0, \varphi_1) = \{(f, g) \in C([0, 1], F_2) \oplus F_1 : f(0) = \varphi_0(g) \text{ and } f(1) = \varphi_1(g)\}.$$

Denote by  $\mathcal{C}$  the class of all unital  $C^*$ -algebras of the form  $A = A(F_1, F_2, \varphi_0, \varphi_1)$  and all finite dimensional  $C^*$ -algebras. These  $C^*$ -algebras are called Elliott-Thomsen building blocks as well as one dimensional non-commutative CW complexes.

Recall that  $\mathcal{C}_0$  is the class of all  $A \in \mathcal{C}$  with  $K_0(A)_+ = \{0\}$  such that  $K_1(A) = 0$ , and  $\mathcal{C}_0^{(0)}$  the class of all  $A \in \mathcal{C}_0$  such that  $K_0(A) = 0$ . Denote by  $\mathcal{C}'_0$  and  $\mathcal{C}_0^{0'}$  the class of all full hereditary  $C^*$ -subalgebras of  $C^*$ -algebras in  $\mathcal{C}_0$  and  $\mathcal{C}_0^{0'}$ , respectively. Recall that  $\mathcal{R}$  denotes the class of finite direct sums of Razak algebras and  $\mathcal{M}_0$  denotes the class of all simple inductive limits of  $C^*$ -algebras in  $\mathcal{R}$  (with injective connecting maps) (see 10. 1, 16.2 and 16. 5 of [17]).

**3.2.** Let  $F_1 = M_{R_1}(\mathbb{C}) \oplus M_{R_2}(\mathbb{C}) \oplus \cdots \oplus M_{R_l}(\mathbb{C})$ , let  $F_2 = M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$  and let  $\varphi_0, \varphi_1 : F_1 \rightarrow F_2$  be (not necessary unital) homomorphisms, where  $R_j$  and  $r_i$  are positive integers. Then  $\varphi_0$  and  $\varphi_1$  induce homomorphisms

$$\varphi_{0*}, \varphi_{1*} : K_0(F_1) = \mathbb{Z}^l \longrightarrow K_0(F_2) = \mathbb{Z}^k$$

by matrices  $(a_{ij})_{k \times l}$  and  $(b_{ij})_{k \times l}$ , respectively, and  $\sum_{j=1}^l a_{ij} R_j \leq r_i$  for  $i = 1, 2, \dots, k$ . We may write  $C([0, 1], F_2) = \bigoplus_{j=1}^k C([0, 1]_j, M_{r_j})$ , where  $[0, 1]_j$  denotes the  $j$ -th interval.

**Lemma 3.3.** Let  $B = A(F_1, F_2, \varphi_0, \varphi_2)$ . Suppose that  $(h, a) \in B_+$  such that  $h_j := h|_{[0, 1]_j}$  has range projection  $P_j$  satisfying the following conditions:

There is a partition  $0 = t_j^0 < t_j^1 < t_j^2 < \cdots < t_j^{n_j} = 1$  such that

- (1) on each open interval  $(t_j^l, t_j^{l+1})$ ,  $P_j(t)$  is continuous and  $\text{rank}(P_j(t)) = r_{j,l}$  is a constant,
- (2) for each  $t_j^l$ ,  $P_j((t_j^l)^+) = \lim_{t \rightarrow (t_j^l)^+} P_j(t)$  (if  $t_j^l < 1$ ) and  $P_j((t_j^l)^-) = \lim_{t \rightarrow (t_j^l)^-} P_j(t)$  (if  $t_j^l > 0$ )

exist,

- (3)  $P_j(t_j^l) \leq P_j((t_j^l)^+)$  and  $P_j(t_j^l) \leq P_j((t_j^l)^-)$

- (4)  $\pi^j(\varphi_0(p)) = P_j(t_j^0) = P_j(0) = P_j(0^+)$  and  $\pi^j(\varphi_1(p)) = P_j(t_j^{n_j}) = P_j(1) = P_j(1^-)$ , where  $p$  is the range projection of  $a \in F_1$ . Then  $\overline{hBh} \in \mathcal{C}$ .



*Proof.* For each closed interval  $[t_j^l, t_j^{l+1}]$ , since

$$P_j((t_j^l)^+) = \lim_{t \rightarrow (t_j^l)^+} P_j(t) \text{ and } P_j((t_j^{l+1})^-) = \lim_{t \rightarrow (t_j^{l+1})^-} P_j(t)$$

exist, we can extend  $P_j|_{(t_j^l, t_j^{l+1})}$  to the closed interval  $[t_j^l, t_j^{l+1}]$ , and denote this projection by  $P_j^l$ . Then we can identify  $P_j^l C([t_j^l, t_j^{l+1}], M_{r_j}) P_j^l \cong C([0, 1], M_{r_{j,l}})$  by identifying  $t_j^l$  with 0 and  $t_j^{l+1}$  with 1, where  $r_{j,l} = \text{rank}(P_j^l)$ . Denote that  $E_2^{j,l} = M_{r_{j,l}}$ . Set  $E_1^{j,l} = P_j(t_j^l) M_{r_j} P_j(t_j^l) \cong M_{R_{j,l}}$ .

Since  $P_j(t_j^l) \leq P_j((t_j^l)^+)$ , we may identify  $E_1^{j,l}$  with a unital hereditary  $C^*$ -subalgebra of  $E_2^{j,l}$ . Denote this identification by  $\psi_0^{j,l} : E_1^{j,l} \rightarrow E_2^{j,l}$ .

Similarly since  $P_j(t_j^l) \leq P_j((t_j^l)^-)$ , we obtain a homomorphism  $\psi_1^{j,l} : E_1^{j,l} \rightarrow E_2^{j,l-1}$  which identifies  $E_1^{j,l}$  with a unital hereditary  $C^*$ -subalgebra of  $E_2^{j,l-1}$ .

Let  $E_1 = pF_1p \oplus_{j=1}^k (\oplus_{l=1}^{n_j-1} E_1^{j,l})$  (note we do not include  $E_1^{j,l}$  for  $l = 0$  and  $l = n_j$ . Instead, we include  $pF_1p$ ) and let  $E_2 = \oplus_{j=1}^k (\oplus_{l=0}^{n_j-1} E_2^{j,l})$ . Let  $\psi_0, \psi_1 : E_1 \rightarrow E_2$  be defined by  $\psi_0|_{pF_1p} = \varphi_0|_{pF_1p} : pF_1p \rightarrow \oplus_{j=1}^k E_2^{j,0}$ ,  $\psi_1|_{pF_1p} = \varphi_1|_{pF_1p} : pF_1p \rightarrow \oplus_{j=1}^k E_2^{j,n_j-1}$ ,  $\psi_0|_{E_1^{j,l}} = \psi_0^{j,l} : E_1^{j,l} \rightarrow E_2^{j,l}$  and  $\psi_1|_{E_1^{j,l}} = \psi_1^{j,l} : E_1^{j,l} \rightarrow E_2^{j,l-1}$ . We then check  $A' = \overline{h'Bh'} \cong A(E_1, E_2, \psi_0, \psi_1) \in \mathcal{C}$ . Namely, each element  $(f, a) = ((f_1, f_2, \dots, f_k), a) \in \overline{h'Bh'}$  corresponds to an element  $(F, b) \in \{C([0, 1], E_2) \oplus E_1 : F(0) = \varphi_0(b), F(1) = \psi_1(b)\}$ , where

$$F = ((f_1^0, f_1^1, \dots, f_1^{n_1-1}, f_2^0, f_2^1, \dots, f_2^{n_2-1}, \dots, f_k^0, f_k^1, \dots, f_k^{n_k-1}) \text{ and } b = (a, f_1(t_1^1), f_1(t_1^2), \dots, f_1(t_1^{n_1-1}), f_2(t_2^1), f_2(t_2^2), \dots, f_2(t_2^{n_2-1}), \dots, f_k(t_k^1), f_k(t_k^2), \dots, f_k(t_k^{n_k-1})))$$

and where  $f_j^l(t) \in E_2^{j,l}$  is defined

$$f_j^l(t) = f_j((t_j^{l+1} - t_j^l)t + t_j^l) \text{ for all } t \in [0, 1], j \in \{1, 2, \dots, k\}, l \in \{0, 1, \dots, n_j - 1\}.$$

□

**3.4.** Let  $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$  be as 3.1. Let  $h = (f, a) \in A_+$  with  $\|h\| = 1$ . For each fixed  $j$ , consider  $f_j = f|_{[0,1]_j}$ . By a simple application of Weyl theorem, one can write eigenvalues of  $f_j(t)$  as continuous function of  $t$

$$\{0 \leq \lambda_{1,j}(t) \leq \lambda_{2,j}(t) \leq \dots \leq \lambda_{r_j,j}(t) \leq 1\}.$$

Let  $e_1, e_2, \dots, e_{r_j}$  be mutually orthogonal rank one projections and put  $f_j' = \sum_{i=1}^{r_j} \lambda_{i,j} e_i$ . Then, on each  $[0, 1]_j$ ,  $f_j$  and  $f_j'$  have exactly the same eigenvalues at each point  $t \in [0, 1]_j$ . Let  $p \in F_1$  be the range projection of  $a \in (F_1)_+$ . By using a unitary in  $C([0, 1]_j, M_{r_j})$ , it is easy to construct a set of mutually orthogonal rank one projections  $p_1, p_2, \dots, p_i, \dots, p_{r_j} \in C([0, 1]_j, M_{r_j})$  such that  $g_j(t) = \sum_{i=1}^{r_j} \lambda_i(t) p_i$  satisfies  $g_j(0) = f_j(0)$  and  $g_j(1) = f_j(1)$ . In particular  $\sum_{\{i, \lambda_i(0) > 0\}} p_i = \pi^j(\varphi_0(p)) \in M_{r_j}$  and  $\sum_{\{i, \lambda_i(1) > 0\}} p_i = \pi^j(\varphi_1(p)) \in M_{r_j}$ , where  $\pi^j : F_2 \rightarrow F_2^j = M_{r_j}$  is the canonical quotient map to the  $j$ -th summand. Then, with  $g|_{[0,1]_j} = g_j$ ,  $(g, a) \in A_+$ . By a result of Thomsen, (see Theorem 1.2 of [50]) (or [43]), for each  $j$  there is a sequence of unitaries  $u_n^j \in C([0, 1]_j, M_{r_j})$  with  $u_n^j(0) = u_n^j(1) = \mathbf{1}_{r_j}$  (Note that as  $g(0) = f(0)$  and  $g(1) = f(1)$ , we can choose  $u_n^j(0) = u_n^j(1) = 1$ ) such that  $g_j = \lim_{n \rightarrow \infty} u_n^j f_j (u_n^j)^*$ . Since  $u_n^j(0) = u_n^j(1) = \mathbf{1}_{r_j}$ , we can put  $u_n^j \in C([0, 1]_j, M_{r_j})$  together to define unitary  $u_n \in \tilde{A}$  and get  $(g, a) = \lim_{n \rightarrow \infty} u_n(f, a) u_n^*$ . In other words,  $(g, a) \sim_{a.u.} (f, a)$  in  $A$ . Note this, in particular, implies that  $\langle (f, a) \rangle = \langle (g, a) \rangle$ .

**Lemma 3.5.** Let  $(g, a) \in A(F_1, F_2, \varphi_0, \varphi_1)_+$  with  $\|(g, a)\| = 1$ . Suppose

$$g_j := g|_{[0,1]_j} = \sum_{i=1}^{r_j} \lambda_{i,j}(t) p_{i,j}(t),$$

where  $\lambda_{i,j} \in C([0,1])_+$  and  $p_{i,j} \in C([0,1], M_{r_j})$  which are mutually orthogonal rank one projections. Then, for any  $\varepsilon > 0$ , there exists  $0 \leq h \leq g$  such that  $\|h - g\| < \varepsilon$ ,  $(h, a) \in A(F_1, F_2, \varphi_0, \varphi_1)$  and  $h_j := h|_{[0,1]_j}$  satisfies the condition described in 3.3.

*Proof.* Fix  $\varepsilon_1 > 0$  and  $j$ . Let  $g_j = g|_{[0,1]_j}$ . Let  $G_{i,j} = \{t \in [0,1] : \lambda_{i,j}(t) = 0\}$ . Since all  $G_{i,j}$  are closed sets, there is  $\delta_0 > 0$  such that if  $0 \notin G_{i,j}$  (or  $1 \notin G_{i,j}$ , respectively), then  $\text{dist}(0, G_{i,j}) > 2\delta_0$  (or  $\text{dist}(1, G_{i,j}) > 2\delta_0$  respectively). Fix  $\delta > 0$  such that  $\delta < \delta_0$ . For each  $i$ , there is closed set  $S_{i,j}$  which is a union of finitely many closed interval containing the set  $G_{i,j}$  such that

$$\text{dist}(s, G_{i,j}) < \delta/4 \text{ for all } s \in S_{i,j}. \quad (\text{e3.1})$$

Hence  $\text{dist}(0, S_{i,j}) > \delta$  (and  $\text{dist}(1, S_{i,j}) > \delta$ ) if  $G_{i,j}$  does not contain them. Choose  $f_{i,j} \in C([0,1])_+$  such that  $f_{i,j}|_{S_{i,j}} = 0$ ,  $1 \geq f_{i,j}(t) > 0$ , if  $t \notin S_{i,j}$  and  $f_{i,j}(t) = 1$  if  $\text{dist}(t, S_{i,j}) > \delta/2$ . Put  $\lambda'_{i,j} = f_{i,j} \lambda_{i,j}$ . Then  $0 \leq \lambda'_{i,j} \leq \lambda_{i,j}$ . Define  $h_j = \sum_{i=1}^{r_j} \lambda'_{i,j} p_{i,j}$ . Then  $h_j \leq g_j$ . We can choose  $\delta$  sufficiently small to begin with so that

$$\|h_j - \sum_{i=1}^{r_j} \lambda_{i,j} p_{i,j}\| < \varepsilon. \quad (\text{e3.2})$$

Put  $h \in C([0,1], F_2)$  such that  $h|_{[0,1]_j} = h_j$ ,  $j = 1, 2, \dots, k$ . Therefore

$$\|h - g\| < \varepsilon. \quad (\text{e3.3})$$

From the construction, we have  $h_j(0) = g_j(0)$  and  $h_j(1) = g_j(1)$  (note that if  $0 \notin G_{i,j}$  (or  $1 \notin G_{i,j}$ ), then  $f_{i,j}(0) = 1$  (or  $f_{i,j}(1) = 1$ )). It follows that  $h(0) = g(0)$  and  $h(1) = g(1)$ . Therefore  $(h, a) \in A(F_1, F_2, \varphi_0, \varphi_1)$ . Moreover,  $(h, a) \leq (g, a)$ .

Let  $q_{i,j}(t) = p_{i,j}(t)$  if  $\lambda'_{i,j}(t) \neq 0$  and  $q_{i,j}(t) = 0$  if  $\lambda'_{i,j}(t) = 0$ . For each  $i$ , there is a partition  $0 = t_{i,j}^{(0)} < t_{i,j}^{(1)} < \dots < t_{i,j}^{(l_j)} = 1$  such that  $q_{i,j}$  is continuous on  $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$ . Namely, on each interval  $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$ ,  $q_{i,j}(t)$  either constant zero projection or rank one projection  $p_{i,j}(t)$  and therefore both  $\lim_{s \rightarrow t_{i,j}^{(l)}+} q_{i,j}(s)$  and  $\lim_{s \rightarrow t_{i,j}^{(l+1)}-} q_{i,j}(s)$  exist. Furthermore, if  $q_{i,j}(t)$  is zero on the open interval  $(t_{i,j}^{(l)}, t_{i,j}^{(l+1)})$ , then  $q_{i,j}(t)$  is also zero on the boundary (since  $\lambda'_{i,j}(t)$  is continuous). Hence we have

$$q_{i,j}((t_{i,j}^{(l)})^+) := \lim_{s \rightarrow t_{i,j}^{(l)}+} q_{i,j}(s) \geq q_{i,j}(t_{i,j}^{(l)}) \quad \text{and} \quad q_{i,j}((t_{i,j}^{(l+1)})^-) := \lim_{s \rightarrow t_{i,j}^{(l+1)}-} q_{i,j}(s) \geq q_{i,j}(t_{i,j}^{(l+1)}).$$

Define  $P_j(t) = \sum_{i=1}^{r_j} q_{i,j}(t)$ . Then  $P_j$  satisfies the conditions described in 3.3.  $\square$

**Theorem 3.6.** Let  $A \in \mathcal{C}'$ . Then  $\text{cer}(u) \leq 2 + \varepsilon$  if  $u \in U_0(A)$ . Moreover, if  $u \in CU(\tilde{A})$  then there exist a continuous path  $\{u(t) : t \in [0,1]\} \subset CU(\tilde{A})$  with  $u(0) = u$ ,  $u(1) = 1_{\tilde{A}}$  and  $\text{cel}(u) \leq 4\pi + \varepsilon$ .

*Proof.* Let  $e \in B := A(F_1, F_2, \varphi_0, \varphi_1)$  with  $\|e\| = 1$  and  $A = \overline{eBe}$ . Let  $u \in U_0(\tilde{A})$  and let  $\varepsilon > 0$ . Without loss of generality, we may assume that  $\varepsilon < \frac{1}{4 \max\{R(i)r_j:i,j\}}$ .

It follows 3.4 that  $e$  is approximately unitarily equivalent (in  $B$ ) to another positive element  $e'$  which has the following form  $e' = (g, a) \in B$  such that

$$g_j := g|_{[0,1]_j} = \sum_{i=1}^{r_j} \lambda_{i,j} p_{i,j}, \quad j = 1, 2, \dots, k. \quad (\text{e3.4})$$



where  $\lambda_{1,j}, \lambda_{2,j}, \dots, \lambda_{r_j,j} \in C([0, 1])$  and mutually orthogonal rank one projections  $p_{1,j}, p_{2,j}, \dots, p_{r_j,j} \in C([0, 1], M_{r_j})$ .

It follows that  $\langle e' \rangle = \langle e \rangle$  in the Cuntz semi-group. Since  $B$  has stable rank one, by [7],  $A$  is isomorphic to  $C := \overline{e'Be'}$ . Therefore, without loss of generality, we may assume that  $u \in \tilde{C}$ . Note that, for any  $f \in C([0, 1]_+)$ ,

$$f(e')|_{[0,1]_j} = \sum_{i=1}^{r_j} f(\lambda_{i,j})p_{i,j}, \quad j = 1, 2, \dots, k. \quad (\text{e3.5})$$

Write  $u = \prod_{i=1}^m \exp(\sqrt{-1}a_i)$ , where each  $a_i = \alpha_i \cdot 1_{\tilde{A}} + x_i$  with  $\alpha_i \in \mathbb{R}$  and  $x_i \in C_{s.a.}$ ,  $i = 1, 2, \dots, m$ . Let  $\delta > 0$ . There is  $1/2 > \eta > 0$  that  $\|f_\eta(e')x_i f_\eta(e') - x_i\| < \delta$ ,  $i = 1, 2, \dots, m$ . To simplify notation, by choosing small  $\delta$ , without loss of generality, we may assume that

$$\|u - \prod_{i=1}^m \exp(\sqrt{-1}\alpha_i \cdot 1_{\tilde{C}} + f_\eta(e')x_i f_\eta(e'))\| < \varepsilon/4. \quad (\text{e3.6})$$

To simplify notation, without loss of generality, we may further assume that  $f_\eta(e')x_i f_\eta(e') = x_i$ ,  $i = 1, 2, \dots, m$ . Let  $\delta_1 > 0$ . It follows from 3.5 that we obtain a  $e'' \leq f_\eta(e')$  such that

$$\|e'' - f_\eta(e')\| < \delta_1 \quad (\text{e3.7})$$

and  $\overline{e''Ce''} \in \mathcal{C}$ . With sufficiently small  $\delta_1$ , we may assume that

$$\|u - \prod_{j=1}^m \exp(i\alpha_j \cdot 1_{\tilde{C}} + e''x_j e'')\| < \varepsilon/3. \quad (\text{e3.8})$$

Put  $v = \prod_{j=1}^m \exp(i\alpha_j \cdot 1_{\tilde{C}} + e''x_j e'')$ . We may now view  $v \in \tilde{D}$ , where  $D = \overline{e''Ce''}$ . Since  $\tilde{D} \in \mathcal{C}$  (see 10.2 of [17]), it follows from 5.19 of [35] that there are  $b_1, b_2 \in \tilde{D}_{s.a.}$  such that  $\|v - \exp(ib_1) \exp(ib_2)\| < \varepsilon/3$ . Note that, if we view  $v \in U_0(\tilde{A})$ ,  $b_1, b_2$  may be viewed as elements in  $\tilde{C}_{s.a.}$  since  $e'' \leq f_\eta(e')$ . This follows that  $\text{cer}(A) \leq 2 + \varepsilon$ .

Now suppose that  $u \in CU(\tilde{A})$ . There exists a commutator unitary  $v \in CU(\tilde{A})$  such that  $\|u - v\| < \varepsilon/4$ . Write  $v = \prod_{s=1}^{m_1} v_s$  and  $v_s = v_{s,1}v_{s,2} \cdots v_{s,r(s)}v_{s,1}^*v_{s,2}^* \cdots v_{s,r(s)}^*$ , where each  $v_{s,i} \in U(\tilde{A})$ ,  $s = 1, 2, \dots, m_1$ . Write  $v_{s,i} = \beta_{s,i} \cdot 1_{\tilde{A}} + z_{s,i}$ , where  $\beta_{s,i} \in \mathbb{C}$  with  $|\beta_{s,i}| = 1$  and  $z_{s,i} \in A$ . For any  $\delta_2 > 0$ , with sufficiently small  $\eta > 0$ , we may assume that

$$\|z_{s,i} - f_\eta(e')z_{s,i}f_\eta(e')\| < \delta_2/16m_1 \left( \sum_{i=1}^{m_1} r(s) \right), \quad 1 \leq i \leq r(s), \quad 1 \leq s \leq m_1. \quad (\text{e3.9})$$

So we may assume that

$$\|z_{s,i} - e''z_{s,i}e''\| < \delta_2/8m_1 \left( \sum_{i=1}^{m_1} r(s) \right), \quad 1 \leq i \leq r(s), \quad 1 \leq s \leq m_1. \quad (\text{e3.10})$$

It follows that there is a unitary in  $w_{s,i} \in \mathbb{C} \cdot 1_{\tilde{A}} + \overline{e''Ae''}$  such that

$$\|v_{s,i} - w_{s,i}\| < \delta_2/4m_1 \left( \sum_{i=1}^{m_1} r(s) \right), \quad 1 \leq i \leq r(s), \quad 1 \leq s \leq m_1. \quad (\text{e3.11})$$

Put  $w_s = w_{s,1}w_{s,2} \cdots w_{s,r(s)}w_{s,1}^*w_{s,2}^* \cdots w_{s,r(s)}^*$  and  $w = \prod_{s=1}^{m_1} w_s$ . With sufficiently small  $\delta_2$ , we may assume that

$$\|w - v\| < \varepsilon/4. \quad (\text{e3.12})$$

Now  $v \in CU(\mathbb{C} \cdot 1_{\tilde{A}} + \overline{e'' A e''})$ . As mentioned above,  $\mathbb{C} \cdot 1_{\tilde{A}} + \overline{e'' A e''} \in \mathcal{C}$ . By 3.16 of [18], in  $\mathbb{C} \cdot 1_{\tilde{A}} + \overline{e'' A e''}$ , there is a continuous path  $\{u(t) : t \in [1/2, 1]\} \subset CU(\mathbb{C} \cdot 1_{\tilde{A}} + \overline{e'' A e''})$  such that  $u(1/2) = w$  and  $u(1) = 1_{\tilde{A}}$  which has the length no more than  $4\pi + \varepsilon/16\pi$ . Note that  $v \in CU(\tilde{A})$  and

$$\|w - u\| < \varepsilon/2, \text{ or } \|uw^* - 1\| < \varepsilon/2. \quad (\text{e } 3.13)$$

Write  $uw^* = \exp(\sqrt{-1}d)$  for some  $d \in \tilde{A}_{s.a.}$ . Then  $\|d\| < 2 \arcsin(\varepsilon/4)$ . Note that  $uw^* \in CU(\tilde{A})$ . Therefore, for each irreducible representation  $\pi$  of  $\tilde{A}_{s.a.}$ ,  $\text{Tr}_\pi(d) = 2m'\pi$  for some integer  $m'$ , where  $\text{Tr}_\pi$  is the standard trace on  $\pi(\tilde{A})$ . Since we choose  $\varepsilon < \frac{1}{4 \max\{R(i)r_j : i, j\}}$ ,  $\text{Tr}_\pi(d) = 0$ . It follows that  $\tau(d) = 0$  for all  $\tau \in T(\tilde{A})$ . Define  $u(t) = \exp(\sqrt{-1}(1-2t)d)w$  for  $t \in [0, 1/2]$ . Note  $u(t)$  is in  $CU(\tilde{A})$  for all  $t \in [0, 1]$  with  $u(0) = u$ ,  $u(1) = 1$  and total length no more than  $4\pi + \varepsilon$ .  $\square$

**3.7.** Let  $A = A(F_1, F_2, \varphi_0, \varphi_1) \in \mathcal{C}$ , where  $F_1 = M_{R_1}(\mathbb{C}) \oplus M_{R_2}(\mathbb{C}) \oplus \cdots \oplus M_{R_l}(\mathbb{C})$ ,  $F_2 = M_{r_1}(\mathbb{C}) \oplus M_{r_2}(\mathbb{C}) \oplus \cdots \oplus M_{r_k}(\mathbb{C})$ . Recall the irreducible representations of  $A$ , are given by

$$\prod_{i=1}^k (0, 1)_i \cup \{\rho_1, \rho_2, \dots, \rho_l\} = \text{Irr}(A),$$

where  $(0, 1)_i$  is the same open interval  $(0, 1)$ . Any trace  $\tau \in TA$  is corresponding to  $(\mu_1, \mu_2, \dots, \mu_k, s_1, s_2, \dots, s_l)$ , where  $\mu_i$  are nonnegative measures on  $(0, 1)_i$  and  $s_j \in \mathbb{R}_+$  and we have

$$\|\tau\| = \sum_{i=1}^k \int_0^1 \mu_i + \sum_{j=1}^l s_j.$$

Let  $t \in (0, 1)_i$  and  $\delta_t$  be the canonical point measure at point  $t$  with measure 1, then

$$\lim_{t \rightarrow 0} \delta_t = (\mu_1, \mu_2, \dots, \mu_k, s_1, s_2, \dots, s_l) \quad \text{and} \quad \lim_{t \rightarrow 1} \delta_t = (\mu_1, \mu_2, \dots, \mu_k, s'_1, s'_2, \dots, s'_l)$$

with  $\mu_j = 0$ ,  $s_j = a_{ij} \cdot \frac{R_j}{r_i}$  and  $s'_j = b_{ij} \cdot \frac{R_j}{r_i}$ , where  $(a_{ij})_{k \times l} = \varphi_{0*}$  and  $(b_{ij})_{k \times l} = \varphi_{1*}$  as in 3.2. Let

$$\lambda = \min_i \left\{ \frac{\sum_{j=1}^l a_{ij} R_j}{r_i}, \frac{\sum_{j=1}^l b_{ij} R_j}{r_i} \right\}.$$

A direct calculation shows that if  $\tau_n \in T(A)$  converge to  $\tau$  in weak\* topology, then  $\|\tau\| \geq \lambda \cdot \limsup \|\tau_n\|$ . In notation of 2.14, we have

$$\lambda_s(A) = \lambda. \quad (\text{e } 3.14)$$

Evidently, the number  $\lambda$  above is the largest positive number satisfying the following conditions

$$\varphi_{0*}([\mathbf{1}_{F_1}]) \geq \lambda \cdot [\mathbf{1}_{F_2}], \quad \varphi_{1*}([\mathbf{1}_{F_1}]) \geq \lambda \cdot [\mathbf{1}_{F_2}] \text{ in } K_0(F_2).$$

In the notation of 2.3, both affine spaces  $\text{Aff}(\tilde{T}(A))$  and  $\text{Aff}(T(A))$  can be identified with the subset of

$$\bigoplus_{j=1}^k C([0, 1]_j, \mathbb{R}) \oplus \mathbb{R}^l = \bigoplus_{j=1}^k C([0, 1]_j, \mathbb{R}) \oplus \underbrace{(\mathbb{R} \oplus \mathbb{R} \oplus \cdots \oplus \mathbb{R})}_{l \text{ copies}}$$

consisting of  $(f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_l)$  satisfying the condition

$$f_i(0) = \frac{1}{r_i} \sum_{j=1}^l a_{ij} g_j \cdot R_j \quad \text{and} \quad f_i(1) = \frac{1}{r_i} \sum_{j=1}^l b_{ij} g_j \cdot R_j.$$

The positive cone  $\text{Aff}(\tilde{T}(A))_+$  is the subset of  $\text{Aff}(\tilde{T}(A))$  consisting all elements of those elements  $(f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_l)$  with  $f_i(t) \geq 0$  and  $g_j \geq 0$  for all  $i, j, t$ . Set  $\mathbb{R}^\sim = \mathbb{R} \cup \{\infty\}$ ,  $\mathbb{R}_+^\sim = \mathbb{R}_+ \cup \{\infty\}$ . Then  $\text{LAff}(\tilde{T}(A))_+$  ( $\text{LAff}^\sim(\tilde{T}(A))$ ), respectively) is identified with the subset of

$$\bigoplus_{j=1}^k LSC([0, 1]_j, \mathbb{R}_+^\sim) \oplus (\mathbb{R}_+^\sim)^l \text{ (or } \bigoplus_{j=1}^k LSC([0, 1]_j, \mathbb{R}^\sim) \oplus (\mathbb{R}^\sim)^l) \quad (\text{e 3.15})$$

consisting of  $(f_1, f_2, \dots, f_k, g_1, g_2, \dots, g_l)$  satisfying the same condition

$$f_i(0) = \frac{1}{r_i} \sum_{j=1}^l a_{ij} g_j \cdot R_j \quad \text{and} \quad f_i(1) = \frac{1}{r_i} \sum_{j=1}^l b_{ij} g_j \cdot R_j.$$

**3.8.** Suppose that  $A = A(F_1, F_2, \varphi_0, \varphi_1)$  is not unital. Let  $e_{i,F_2} = (e_{i,2}, e_{i,2}, \dots, e_{i,k}) \in F_2$  be a projection such that  $\mathbf{1}_{F_2} - \varphi_i(\mathbf{1}_{F_1}) = e_{i,F_2}$ ,  $i = 0, 1$ . Put  $F_{2,i} = e_{i,F_2} F_2 e_{i,F_2}$ ,  $i = 0, 1$ . Define  $\varphi'_i : \mathbb{C} \rightarrow F_{2,i}$  by  $\varphi'_i(\lambda) = \lambda e_{i,F_2}$ ,  $i = 1, 2$ . Define  $F_1^\sim = F_1 \oplus \mathbb{C}$  and  $\varphi_i^\sim : F_1^\sim \rightarrow F_2$  by  $\varphi_i^\sim(a \oplus \lambda) = \varphi_i(a) \oplus \lambda e_{i,F_2}$ ,  $i = 0, 1$ . Then  $\tilde{A} = A(F_1^\sim, F_2, \varphi_0^\sim, \varphi_1^\sim)$ .

In what follows, we will use notations  $\mathbb{Z}^\sim = \mathbb{Z} \cup \{\infty\}$ , and  $\mathbb{Z}_+^\sim = \mathbb{Z}_+ \cup \{\infty\}$ . Let  $B = A(F_1, F_2, \varphi_0, \varphi_1)$ . Let  $a \in B_+$ , define  $r_a \in \text{LAff}(\tilde{T}(A))_+$  by  $r_a(\tau) = d_\tau(a) = \lim_{n \rightarrow \infty} \tau(a^{1/n})$ . When one identifies  $\text{LAff}(\tilde{T}(A))_+$  with the subspace of  $\bigoplus_{j=1}^k LSC([0, 1]_j, \mathbb{R}_+^\sim) \oplus (\mathbb{R}_+^\sim)^l$  as in 3.7,  $r_a \in \bigoplus_{j=1}^k LSC([0, 1]_j, \frac{1}{r_j} \mathbb{Z}_+^\sim) \oplus \bigoplus_{i=1}^l (\frac{1}{R_i} \mathbb{Z}_+^\sim)$ . (Recall that map  $\varphi_{i,*} : K_0(F_1) = \mathbb{Z}^l \rightarrow K_0(F_2) = \mathbb{Z}^k$  ( $i = 0, 1$ ), induced by  $\varphi_i : F_1 \rightarrow F_2$  is given by the matrix  $(a_{ij})_{k \times l}$  and  $(b_{ij})_{k \times l}$  with nonnegative integer entries, which can be extended to maps (still denoted by  $\varphi_{i,*}$ ) from  $(\mathbb{Z}^\sim)^l$  to  $(\mathbb{Z}^\sim)^k$ .) If we identify each  $\frac{1}{r_j} \mathbb{Z}$  (or  $\frac{1}{R_i} \mathbb{Z}$  respectively) with  $\mathbb{Z}$  by identifying  $\frac{1}{r_j}$  with  $1 \in \mathbb{Z}$  (or by identifying  $\frac{1}{R_i}$  with  $1 \in \mathbb{Z}$ ),  $r_a$  is identified with

$$((f_1, f_2, \dots, f_k), (j_1, j_2, \dots, j_l)) \in \bigoplus_{j=1}^k LSC([0, 1]_j, \mathbb{Z}_+^\sim) \oplus (\mathbb{Z}_+^\sim)^l$$

which satisfy

$$(f_1(0), f_2(0), \dots, f_k(0)) = \varphi_{0,*}(j_1, j_2, \dots, j_l) \text{ and } (f_1(1), f_2(1), \dots, f_k(1)) = \varphi_{1,*}(j_1, j_2, \dots, j_l).$$

Let  $LSC([0, 1], \mathbb{R}^\sim)$  be the set of lower-semicontinuous functions from  $[0, 1]$  to  $\mathbb{R}^\sim$ . We will use the notation  $LSC([0, 1], (\mathbb{R}^\sim)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{R}^\sim)^l$  to denote the subset of  $LSC([0, 1], (\mathbb{R}^\sim)^k) \oplus (\mathbb{R}^\sim)^l$  consisting of elements  $((f_1, f_2, \dots, f_k), (j_1, j_2, \dots, j_l)) \in LSC([0, 1], (\mathbb{R}^\sim)^k) \oplus (\mathbb{R}^\sim)^l$  satisfying

$$(f_1(0), f_2(0), \dots, f_k(0)) = \varphi_{0,*}(j_1, j_2, \dots, j_l) \text{ and } (f_1(1), f_2(1), \dots, f_k(1)) = \varphi_{1,*}(j_1, j_2, \dots, j_l).$$

Let  $LSC([0, 1], (\mathbb{R}_+^\sim)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{R}_+^\sim)^l$  ( $LSC([0, 1], (\mathbb{Z}^\sim)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}^\sim)^l$ , or  $LSC([0, 1], (\mathbb{Z}_+^\sim)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+^\sim)^l$  respectively) be the subset of  $LSC([0, 1], (\mathbb{R}^\sim)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{R}^\sim)^l$  consisting of the above elements with  $f_i(t)$  and  $j_i \in \mathbb{R}_+^\sim$  ( $\in \mathbb{Z}^\sim$  or  $\in \mathbb{Z}_+^\sim$  respectively). If we insist not take the value  $+\infty$ , then we will use the notation  $LSC_f$  instead of  $LSC$ . So the sets  $LSC_f([0, 1], (\mathbb{R}_+)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{R}_+)^l$  and  $LSC_f([0, 1], (\mathbb{Z}_+)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+)^l$  can also be defined similarly.

Now let  $B \in \mathcal{C}_0$ . Let  $C \in \mathcal{C}'$  be a full hereditary subalgebra of  $B$ . Using the rank function in 3.17 of [18] and applying 3.18 of [18], The map  $r : \langle a \rangle \mapsto r_a$  gives an injective semi-group homomorphism from  $W(C)$  to  $LSC_f([0, 1], (\mathbb{Z}_+)^k) \oplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+)^l$  (see also 3.18 of [18]) which extends to an order injective semi-group homomorphism from  $Cu(C)$  to

$LSC([0, 1], (\mathbb{Z}_+^\sim)^k) \bigoplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+^\sim)^l$ . Note  $\tilde{C} \in \mathcal{C}$ . Also note that  $Cu^\sim(C)$  (see [43]) is the semi-group of the formal differences  $f - n[1_{\tilde{C}}]$ , with  $n \in \mathbb{Z}_+$  and  $f \in Cu(\tilde{C})$  such that  $Cu(\pi_C)(f) = [n]$ , where  $Cu(\pi_C)$  is the map induced by the quotient map  $\pi_C : \tilde{C} \rightarrow \mathbb{C}$ . With the help of discussion of 3.4, it is straight forward to check the following:

**Proposition 3.9.** *Let  $C \in C'_0$ . Then*

$$W(C) = LSC_f([0, 1], (\mathbb{Z}_+)^k) \bigoplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+)^l \text{ and} \quad (\text{e 3.16})$$

$$Cu(C) = LSC([0, 1], (\mathbb{Z}_+^\sim)^k) \bigoplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}_+^\sim)^l. \quad (\text{e 3.17})$$

Moreover (see [43] for the definition of  $Cu^\sim$ )

$$Cu^\sim(C) = LSC([0, 1], (\mathbb{Z}^\sim)^k) \bigoplus_{(\varphi_{0,*}, \varphi_{1,*})} (\mathbb{Z}^\sim)^l. \quad (\text{e 3.18})$$

Since  $C$  is stably projectionless, it follows that the order  $Cu^\sim(C)$  is determined by  $Cu(C)$ .

**Definition 3.10.** Fix an integer  $a_1 \geq 1$ . Let  $\alpha = \frac{a_1}{a_1+1}$ . For each  $r \in \mathbb{Q}_+ \setminus \{0\}$ , let  $e_r \in Q$  be a projection with  $\text{tr}(e_r) = r$ . Let  $\tilde{Q}_r := (1 \otimes e_r)(Q \otimes Q)(1 \otimes e_r)$ . Define  $q_r : Q \rightarrow \tilde{Q}_r$  by  $a \mapsto a \otimes e_r$  for  $a \in Q$ . We will also use  $q_r$  to denote any homomorphism from  $B$  to  $B \otimes e_r Q e_r$  (or to  $B \otimes Q$ ) defined by sending  $b \in B$  to  $b \otimes e_r \in B \otimes e_r Q e_r \subset B \otimes Q$ .

For  $r = \alpha = \frac{a_1}{a_1+1}$ , one can identify  $Q$  with  $Q \otimes M_{a_1+1}$ , then the projection  $e_\alpha$  is identified with  $\mathbf{1}_Q \otimes \text{diag}(\underbrace{1, \dots, 1}_{a_1}, 0)$ .

Let

$$R(\alpha, 1) = \{(f, a) \in C([0, 1], Q \otimes Q) \oplus Q : f(0) = q_\alpha(a) \text{ and } f(1) = a \otimes \mathbf{1}_Q\}.$$

Note that an element  $(f, a)$  is full in  $R(\alpha, 1)$  if and only if  $a \neq 0$  and  $f(t) \neq 0$  for all  $t \in (0, 1)$ . Let  $a_\alpha = (f, 1)$  be defined as follows. Let

$$f(t) = (1 - t)(1 \otimes e_\alpha) + t(1 \otimes 1) \text{ for all } t \in (0, 1). \quad (\text{e 3.19})$$

Note that  $a_\alpha$  is a *strictly positive element* of  $R(\alpha, 1)$ , moreover, for any  $1/2 > \eta > 0$ ,  $f_\eta(a_\alpha)$  is full.  $C^*$ -algebra  $R(\alpha, 1)$  and  $a_\alpha$  will appear frequently in this paper.

Let  $LSC([0, 1], \mathbb{R}^\sim) \oplus_\alpha \mathbb{R}^\sim$  (or  $LSC_f([0, 1], \mathbb{R}_+) \oplus_\alpha \mathbb{R}_+$  respectively) be the subset of  $LSC([0, 1], \mathbb{R}^\sim) \oplus \mathbb{R}^\sim$  (or  $LSC_f([0, 1], \mathbb{R}_+) \oplus \mathbb{R}_+$  respectively) consisting of elements  $(f, x)$  such that  $f(0) = \alpha x$  and  $f(1) = x$ . The rank function  $r : \langle a \rangle \mapsto r(a) = d_\tau(a)$  gives maps from  $W(R(\alpha, 1))$  to  $LSC_f([0, 1], \mathbb{R}_+) \oplus_\alpha \mathbb{R}_+$  and from  $Cu(R(\alpha, 1))$  to  $LSC([0, 1], \mathbb{R}_+^\sim) \oplus_\alpha \mathbb{R}_+^\sim$  which are order semi-group homomorphisms. But these maps are only surjective but not injective.

Recall that  $W(Q)$  and  $Cu(Q)$  can be identified with the semi-groups  $\mathbb{R}_+ \setminus \{0\} \sqcup \mathbb{Q}_+$  and  $\mathbb{R}_+^\sim \setminus \{0\} \sqcup \mathbb{Q}_+$ , where the second copy of  $\mathbb{Q}$  is identified with  $K_0(Q)$  and  $\mathbb{R}_+^\sim \setminus \{0\}$  identified with the rank functions of non-projection and non-zero positive elements. With the order in  $Cu(Q)$ , in  $\mathbb{R}^\sim \sqcup \mathbb{Q}$ ,  $t < [t]$  for  $r \in \mathbb{Q} \subset \mathbb{R}$  and  $[t] \in K_0(Q) = \mathbb{Q}$ . But  $s > [t]$  if  $s > t$  as in  $\mathbb{R}^\sim$ .

A function  $f : [0, 1] \rightarrow \mathbb{R}^\sim \sqcup \mathbb{Q}$  is called lower-semicontinuous if, for each  $t_0 \in [0, 1]$ , and if  $f(t_0) = [r] \in K_0(Q)$ , there exists  $\delta > 0$  such that  $f(t) \geq f(t_0)$  for all  $t \in (t_0 - \delta, t_0 + \delta) \cap [0, 1]$ , or, if  $f(t_0) = r \in \mathbb{R}^\sim$ , for any non zero  $r \in \mathbb{R}_+^\sim \setminus \{0\} \sqcup \mathbb{Q}_+$ , there exists  $\delta > 0$  such that

$$f(t) + r \geq f(t_0) \text{ for all } t \in [0, 1] \cap (t_0 - \delta, t_0 + \delta) \setminus \{t_0\},$$

where the order is in  $\mathbb{R}^\sim \sqcup \mathbb{Q}$  mentioned above.

Let  $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q})$  be the set of all lower-semicontinuous functions.

Let  $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus_\alpha \mathbb{R}^\sim \sqcup \mathbb{Q}$  be the subset of  $LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus \mathbb{R}^\sim \sqcup \mathbb{Q}$  consisting of elements  $(f, x)$  such that  $f(0) = \alpha x$  and  $f(1) = x$ . (Here we define  $\alpha[r] = [\alpha r]$ ). The sets  $LSC([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+$  and  $LSC_f([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+$  can be defined similarly. Then we have the following fact.

**Corollary 3.11.** *Let  $A = R(\alpha, 1)$  for some  $1 > \alpha > 0$ . Then*

$$W(A) = LSC_f([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+, \quad (\text{e 3.20})$$

$$Cu(A) = LSC([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+ \text{ and} \quad (\text{e 3.21})$$

$$Cu^\sim(A) = LSC([0, 1], \mathbb{R}^\sim \sqcup \mathbb{Q}) \oplus_\alpha \mathbb{R}^\sim \sqcup \mathbb{Q}. \quad (\text{e 3.22})$$

Note, with (e 3.22), map  $r$  can be extended to an order semi-group homomorphism from  $Cu^\sim(A)$  to  $LSC([0, 1], \mathbb{R}^\sim) \oplus_\alpha \mathbb{R}^\sim$  defined by  $r(f(s), a) = (r(f(s)), r(a))$ , where  $r(t) = t$  for all  $t \in \mathbb{R}^\sim$  and  $r([t]) = t$  for all  $t \in \mathbb{Q}$ .

**Definition 3.12.** (cf. 12.4, 12.1 and 15.6 of [17]) We would like to recall the definition of class  $\mathcal{D}$  and  $\mathcal{D}_0$ .

Let  $A$  be a non-unital simple  $C^*$ -algebra with a strictly positive element  $a \in A$  with  $\|a\| = 1$ . Suppose that there exists  $1 > f_a > 0$ , for any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subset A$  and any  $b \in A_+ \setminus \{0\}$ , there are  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear maps  $\varphi : A \rightarrow A$  and  $\psi : A \rightarrow D$  for some  $C^*$ -subalgebra  $D \subset A$  with  $D \in \mathcal{C}_0^{0'}$  (or  $\mathcal{C}_0'$ )

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F} \cup \{a\}, \quad (\text{e 3.23})$$

$$\varphi(a) \lesssim b, \quad (\text{e 3.24})$$

$$t(f_{1/4}(\psi(a))) \geq f_a \text{ for all } t \in T(D). \quad (\text{e 3.25})$$

Then we say  $A \in \mathcal{D}_0$  (or  $\mathcal{D}$ ). Note that we assume that  $D \perp \overline{\varphi(A)A\varphi(A)}$ .

Let  $A$  be a separable stably projectionless simple  $C^*$ -algebra. We say  $A$  has generalized tracial rank at most one and write  $gTR(A) \leq 1$ , if there exists  $e \in P(A)_+$  with  $\|e\| = 1$  such that  $\overline{eAe} \in \mathcal{D}$ .

**Definition 3.13.** Let  $A \in \mathcal{D}$  as defined 3.12. If in addition, for any integer  $n$ ,  $D = M_n(D_1)$  for some  $D_1 \in \mathcal{C}_0'$  such that

$$\psi(x) = \text{diag}(\overbrace{\psi_1(x), \psi_1(x), \dots, \psi_1(x)}^n) \text{ for all } x \in \mathcal{F}, \quad (\text{e 3.26})$$

where  $\psi_1 : A \rightarrow D_1$  is an  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map, then we say  $A \in \mathcal{D}^d$ .

It follows from 14.5 of [17] that, if  $A \in \mathcal{D}_0$ , then  $A \in \mathcal{D}^d$ . Moreover,  $D_1$  can be chosen in  $\mathcal{C}_0^{0'}$ .

**Proposition 3.14.** *Let  $A \in \mathcal{D}$  with continuous scale and let  $e \in A_+$  with  $\|e\| = 1$  be a strictly positive element and  $1 > f_e > 0$  be as in 3.12. Then, for any finite subset  $\mathcal{F} \subset A$ , any  $\varepsilon > 0$ , any  $b \in A_+ \setminus \{0\}$  and any integer  $n \geq 1$ , there are  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear maps  $\varphi : A \rightarrow A$  and  $\psi : A \rightarrow M_n(D)$  for some  $C^*$ -subalgebra  $D \subset A$  with  $D \in \mathcal{C}_0'$  such that*

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F} \cup \{e\}, \quad (\text{e 3.27})$$

$$\varphi(e) \lesssim b, \quad (\text{e 3.28})$$

$$t(f_{1/4}(\psi(e))) \geq f_e/2 \text{ for all } t \in T(D). \quad (\text{e 3.29})$$

*Proof.* Fix  $\varepsilon > 0$ ,  $b$  and  $\mathcal{F}$  as described in the statement. Let  $\eta = \inf\{\tau(b) : \tau \in \overline{T(A)}^w\} > 0$ . Choose  $e_0 \in A_+$  with  $\|e_0\| = 1$  such that  $\|e_0 e e_0 - e\| < \varepsilon/16$ . Without loss of generality, we may also assume that  $e_0 f = f e_0 = f$  for all  $f \in \mathcal{F}$ . It follows from 15.8 of [17] that the map from  $W(A)$  to  $\text{LAff}_{b+}(\overline{T(A)}^w)$  is surjective. Therefore, without loss of generality, we may also write that  $e_0 = \sum_{i=1}^n e_{0,i}$ , where  $\{e_{0,1}, e_{0,2}, \dots, e_{0,n}\}$  are mutually orthogonal and there exists  $w_i \in A$  such that  $w_i^* w_i = e_{0,1}$  and  $w_i w_i^* = e_{0,i}$ ,  $i = 1, 2, \dots, n$ . There is a finite subset  $\{x_1, x_2, \dots, x_m\} \subset A$  such that

Since  $A$  is stably projectionless, without loss of generality, we may assume that  $sp(e_0) = [0, 1]$ . Then elements  $e_{0,i}$  and  $w_i$  generate a  $C^*$ -subalgebra  $C$  which is isomorphic to  $C_0((0, 1]) \otimes M_n$  which is semi-projective. Let  $\mathcal{G}_1 = \{e_i, w_i : 1 \leq i \leq n\}$ .

Put  $\delta_0 = \min\{\varepsilon/16(n+1), \eta/2(n+1), f_e/(4(n+1))\}$ .

Choose  $\delta_1 > 0$  such that for any  $\mathcal{G}_1$ - $\delta_1$ -multiplicative completely positive contractive linear map  $L$  from  $C$  to a  $C^*$ -algebra  $B$  there is a homomorphism  $\varphi' : C \rightarrow B$  such that

$$\|\varphi'(g) - L(g)\| < \delta_0/4m \text{ for all } g \in \mathcal{G}_1. \quad (\text{e 3.30})$$

Put  $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_1 \cup \{ab : a, b \in \mathcal{F} \cup \mathcal{G}_1\}$ .

Fix a positive number  $\varepsilon_1 < \min\{\delta_0, \delta_1/2\}/(4(n+1))$ . Since  $A \in \mathcal{D}$ , there are  $\mathcal{F}_2$ - $\varepsilon_1$ -multiplicative completely positive contractive linear maps  $\varphi : A \rightarrow A$  and  $\psi_0 : A \rightarrow B$  for some  $C^*$ -subalgebra  $B \subset A$  with  $B \in \mathcal{C}'_0$  such that  $\varphi(e) \lesssim b$ ,

$$\|x - \text{diag}(\varphi(x), \psi_0(x))\| < \varepsilon_1 \text{ for all } x \in \mathcal{F}_1 \cup \{e, e_0\}, \quad (\text{e 3.31})$$

$$t(f_{1/4}(\psi(e))) \geq f_e \text{ for all } t \in T(B). \quad (\text{e 3.32})$$

By the choice of  $\mathcal{G}_1$  and  $\delta_1$ , we obtain a homomorphism  $h : C \rightarrow B$  such that

$$\|h(g) - \psi_0(g)\| < \delta_0/4 \text{ for all } g \in \mathcal{G}_1. \quad (\text{e 3.33})$$

Let  $e'_i = h(e_i)$  and  $v_i = h(w_i)$ ,  $i = 1, 2, \dots, n$ . Let  $B' = h(e_0)Bh(e_0)$ . Since  $h$  is a homomorphism and  $e', v_i \in B'$ ,  $B' \cong M_n(\overline{e'_1 B e'_1})$ . Set  $D = \overline{e'_1 B e'_1}$ . Define  $\psi : A \rightarrow B'$  by  $\psi(a) = h(e_0)\psi(a)h(e_0)$ . One checks

$$\tau(\psi(e)) \geq f_a/2 \text{ for all } \tau \in T(B') \quad (\text{e 3.34})$$

and  $\psi$  is  $\mathcal{F}$ - $\varepsilon$ -multiplicative. Moreover,

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}. \quad (\text{e 3.35})$$

□

## 4 The unitary group

**Lemma 4.1.** *Let  $A$  be a non-unital  $C^*$ -algebra and let  $e_1, e_2 \in A_+$  with  $\|e_i\| = 1$  ( $i = 1, 2$ ) such that*

$$e_1 e_2 = e_2 e_1 = 0$$

*and there is a unitary  $u \in \tilde{A}$  such that  $u^* e_1 u = e_2$ . Suppose that  $w = 1_{\tilde{A}_0} + x_0 \in \tilde{A}_0$  is a unitary with  $x_0 \in A_0$ , where  $A_0 = \overline{e_1 A e_1}$ . Then  $w_1 = 1 + x_0 + u^* x_0^* u \in CU(\tilde{A})$ ,  $\text{cel}(w_1) \leq \pi$  and  $\text{cer}(w_1) \leq 1 + \varepsilon$ .*



*Proof.* Let  $B$  be the  $C^*$ -subalgebra of  $A$  generated by  $A_0, e_2u$ . Note that  $u^*A_0u = \overline{e_2Ae_2}$ . Therefore  $B \cong M_2(A_0)$ . Consider  $M_2(\tilde{A}_0)$ . Put  $p_{1,1} = 1_{\tilde{A}_0}$ . We view  $p_{1,1}$  as the open projection associated to  $A_0$ . Let  $p_{2,2} = u^*p_{1,1}u$ . Define, for  $t \in [0, 1]$ ,

$$X(t) = ((\cos(t\pi/2))p_{1,1} + (\sin(t\pi/2))p_{1,1}u + (\sin(t\pi/2))u^*p_{1,1} + (\cos(t\pi/2))p_{2,2}) + ((1_{\tilde{A}}) - p_{1,1} - p_{2,2}).$$

Define

$$W(t) = (1 + x_0)X(t)^*(1 + x_0^*)X(t)^* \text{ for all } t \in [0, 1].$$

Let  $X'(t) = X(t) - ((1_{\tilde{A}}) - p_{1,1} - p_{2,2}) \in M_2(\tilde{A}_0)$  and

$$W'(t) = (p_{1,1} + p_{2,2} + x_0)X'(t)(p_{1,1} + p_{2,2} + x_0^*)X'(t)^* \in M_2(\tilde{A}_0).$$

We have

$$X'(0) = p_{1,1} + p_{2,2} \text{ and } X'(1) = p_{1,1}u + u^*p_{1,1}.$$

Then

$$W'(0) = p_{1,1} + p_{2,2} \text{ and } W'(1) = (p_{1,1} + x_0) + (p_{2,2} + u^*x_0^*u).$$

Let  $\pi : M_2(\tilde{A}_0) \rightarrow \widetilde{M_2(A)}$  be the quotient map. Then  $\pi(W'(t)) = 1_{M_2}$  for all  $t \in [0, 1]$ . This implies that  $W'(t) \in \widetilde{M_2(A)}$  for all  $t \in [0, 1]$ . It follows that  $W(t) \in U(\tilde{A})$  for all  $t \in [0, 1]$ . Note that  $W(0) = 1_{\tilde{A}}$  and  $W(1) = 1 + x_0 + u^*x_0^*u$ . Moreover, one computes that (since each  $W(t) \in U_0(\tilde{A})$ ),

$$\text{cel}(\{W(t)\}) \leq \pi.$$

It follows that  $\text{cer}(W(1)) \leq 1 + \varepsilon$ . Moreover

$$1 + x_0 + u^*x_0^*u = (1 + x_0)u^*(1 + x_0^*)u.$$

It follows that  $1 + x_0 + u^*x_0^*u \in CU(\tilde{A})$ . □

The following is a variation of a lemma of N. C. Phillips

**Lemma 4.2** (Lemma 3.1 of [32]). *Let  $H > 0$  be a positive number and let  $N \geq 2$  be an integer. Then, for any non-unital  $C^*$ -algebra which has almost stable rank one, any positive element  $e_0 \in A_+$  with  $\|e_0\| = 1$ , and  $u = \lambda \cdot 1_{\tilde{A}_0} + x_0 \in \tilde{A}_0$  (where  $x_0 \in A_0$  and  $|\lambda| = 1$ ) such that  $\text{cel}_{\tilde{A}_0}(u) \leq H$ , where  $A_e = \overline{e_0Ae_0}$ . Suppose that there are mutually orthogonal positive elements  $e_1, e_2, \dots, e_{2N} \in A_e^\perp$  such that  $e_0 \sim e_i$ ,  $i = 1, 2, \dots, 2N$ . Then*

$$\|u' - \lambda \cdot z\| < 2H/N,$$

where  $u' = \lambda \cdot 1_{\tilde{A}} + x_0$  and  $z \in CU(\tilde{A})$  with  $\text{cel}(z) \leq 2\pi$  and  $\text{cer}(z) \leq 2 + \varepsilon$ .

*Proof.* Since  $\text{cel}_{\tilde{A}_0}(u) \leq H$ , there are  $u_0, u_1, \dots, u_N \in \tilde{A}_0$  such that

$$u_0 = u, \quad u_N = 1_{\tilde{A}_0} \text{ and } \|u_i - u_{i-1}\| < H/N, \quad i = 1, 2, \dots, N. \quad (\text{e4.1})$$

Write  $u_i = \lambda_i \cdot 1_{\tilde{A}_0} + x'_i$ , where  $x'_i \in A_0$ ,  $i = 1, 2, \dots, N$ . It follows from (e4.1) that

$$|\lambda_i - \lambda_{i-1}| < H/N, \quad i = 1, 2, \dots, N.$$

Let  $v = v_0 = \bar{\lambda}u = 1_{\tilde{A}_0} + \bar{\lambda}x'_0$  and  $v_i = \bar{\lambda}_i u_i = 1_{\tilde{A}_0} + \bar{\lambda}_i x'_i$ ,  $i = 1, 2, \dots, N$ . Put  $x_i = \bar{\lambda}_i x'_i$ ,  $i = 0, 1, \dots, N$ . Now

$$\|v_i - v_{i-1}\| = \|\bar{\lambda}_i u_i - \bar{\lambda}_{i-1} u_{i-1}\| < 2H/N, \quad i = 1, 2, \dots, N. \quad (\text{e4.2})$$

Let

$$\varepsilon_0 = 2H/N - \sup\{\|v_i - v_{i-1}\| : i = 1, 2, \dots, N\}.$$

Choose  $1 > \delta > 0$  such that

$$\|x_i - f_\delta(e_0)x_i f_\delta(e_0)\| < \varepsilon_0/16N, \quad i = 0, 1, 2, \dots, N.$$

Put  $B_0 = \overline{f_\delta(e_0)A f_\delta(e_0)}$ . There is a unitary  $w_i \in 1_{\tilde{A}_0} + B_0$  such that

$$\|v_i - w_i\| < \varepsilon_1/8N, \quad i = 0, 1, \dots, N.$$

Write  $w_i = 1_{\tilde{A}_0} + y_i$ , where  $y_i \in B_0$ . Since  $A$  has almost stable rank one, there are unitaries  $U_i \in \tilde{A}$  such that

$$U_i^* f_{\delta/2}(e_0) U_i \in \overline{e_i A e_i}, \quad i = 1, 2, \dots, 2N.$$

Let

$$X_1 = 1_{\tilde{A}} + y_0 + \sum_{i=1}^N U_{2i-1}^* y_i^* U_{2i-1} + \sum_{i=1}^N U_{2i}^* y_i U_{2i} \quad (\text{e 4.3})$$

$$X_2 = 1_{\tilde{A}} + y_0 + \sum_{i=1}^N U_{2i-1} y_{i-1}^* U_{2i-1} + \sum_{i=1}^N U_{2i}^* y_{i-1} U_{2i} \quad \text{and} \quad (\text{e 4.4})$$

$$X_3 = 1_{\tilde{A}} + y_0 + \sum_{i=1}^N U_{2i-1}^* y_i U_{2i-1} + \sum_{i=1}^N U_{2i}^* y_i^* U_{2i}. \quad (\text{e 4.5})$$

Note that  $X_1 \in U(\tilde{A})$ . As in 4.1,

$$X_i \in CU(\tilde{A}), \quad \text{cel}(X_i) \leq \pi \quad \text{and} \quad \text{cer}(X_i) \leq 1 + \varepsilon, \quad i = 2, 3. \quad (\text{e 4.6})$$

Moreover

$$\|X_1 - X_2\| < \varepsilon_0/8N + \sup\{\|v_i - v_{i-1}\| : i = 1, 2, \dots, N\} \quad (\text{e 4.7})$$

Furthermore,

$$1_{\tilde{A}} + y_0 = X_1 X_3. \quad (\text{e 4.8})$$

Put  $z = X_2 X_3$ . Then, by (e 4.6),

$$z \in CU(\tilde{A}), \quad \text{cel}(z) \leq 2\pi \quad \text{and} \quad \text{cer}(z) \leq 2 + \varepsilon.$$

Moreover,

$$\|\bar{\lambda} \cdot u' - z\| \leq \|(1_{\tilde{A}} - 1_{\tilde{A}_0}) + v_0 - (1_{\tilde{A}} + y_0)\| + \|(1_{\tilde{A}} + y_0) - z\| \quad (\text{e 4.9})$$

$$< \varepsilon_0/16N + \varepsilon_0/8N + \sup\{\|v_i - v_{i-1}\| : i = 1, 2, \dots, N\} < 2H/N. \quad (\text{e 4.10})$$

□

**Theorem 4.3.** (cf. Theorem 6.5 of [29]) *Let  $A$  be a non-unital separable simple  $C^*$ -algebra in  $\mathcal{D}$  and let  $u \in U_0(\tilde{A})$  with  $u = \lambda \cdot 1 + x_0$ , where  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$  and  $x_0 \in A$ . Then, for any  $\varepsilon > 0$ , there exists a unitary  $u_1, u_2 \in \tilde{A}$  such that  $u_1$  has exponential length no more than  $2\pi$ ,  $u_2$  has exponential rank 3 and*

$$\|u - u_1 u_2\| < \varepsilon.$$

Moreover,  $\text{cer}(A) \leq 5 + \varepsilon$ .

*Proof.* Let  $1/2 > \varepsilon > 0$ . Let  $u' = \bar{\lambda} \cdot u$ . Let  $v_0, v_1, \dots, v_n \in U_0(\tilde{A})$  such that

$$v_0 = u', \quad v_n = 1 \quad \text{and} \quad \|v_i - v_{i-1}\| < \varepsilon/16, \quad i = 0, 1, \dots, n-1.$$

Write  $v_i = \lambda_i \cdot 1 + x_i$ , where  $|\lambda_i| = 1$  and  $x_i \in A$ ,  $i = 0, 1, \dots, n$ . Note that  $x_n = 0$ .

As demonstrated in the proof of 4.2, we may assume that there is a strictly positive element  $e \in A_+$  such that  $\|e\| = 1$  such that

$$f_\eta(e)x_i = x_i f_\eta(e) = x_i, \quad i = 0, 1, 2, \dots, n, \quad (\text{e 4.11})$$

for some  $\eta > 0$ . Let

$$\mathcal{G}_1 = \{e, f_\eta(e), f_{\eta/2}(e), x_i, 0 \leq i \leq n\}.$$

Put

$$d = \inf\{d_\tau(e) : \tau \in \overline{T(A)}^w\} > 0.$$

Without loss of generality, we may assume that  $\tau(f_{1/2}(e)) \geq d/2$  for all  $\tau \in \overline{T(A)}^w$ .

Let  $\delta > 0$  and let  $\mathcal{G} \supset A$  be a finite subset. Let  $e'_0 \in A_+ \setminus \{0\}$  such that

$$d_\tau(e'_0) < d/4(n+1) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

Since  $A \in \mathcal{D}$ , there are  $A_0$  and  $M_n(D) \subset A$  with  $D \in \mathcal{C}'_0$ ,  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps  $\varphi_0 : A \rightarrow A_0$  and  $\varphi_1 : A \rightarrow M_n(D)$ , such that

$$\|x - \text{diag}(\varphi_0(x), \varphi_1(x))\| < \delta \quad \text{for all } x \in \mathcal{G} \quad (\text{e 4.12})$$

$$\varphi_0(e) \lesssim e'_0, \quad (\text{e 4.13})$$

$$\tau(f_{1/4}(\varphi_1(e))) \geq d/4 \quad \text{for all } \tau \in \overline{T(A)}^w. \quad (\text{e 4.14})$$

By choosing small  $\delta$  and large  $\mathcal{G}$ , we may assume the following: there are  $y_i \in \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$  such that  $\lambda_i \cdot 1 + y_i$  is a unitary,

$$\|y_i - y_{i+1}\| < \varepsilon/8, \quad \|(\lambda_i \cdot 1 + y_i) - (\lambda_i \cdot 1 + y_{i-1})\| < \varepsilon/8, \quad (\text{e 4.15})$$

$i = 0, 1, \dots, n$ , and there are  $z_1 \in \overline{M_n(f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e)))}$  such that  $1_{\tilde{A}} + z_1$  is a unitary such that

$$\|v_0 - (1_{\tilde{A}} + y_0 + z_1)\| < \varepsilon/8 \quad \text{and} \quad y_n = 0. \quad (\text{e 4.16})$$

Put  $u_1 = 1 + y_0$ ,  $u'_2 = 1 + z_1$  and  $u_2 = \lambda u'_2$ . Then

$$\|u - u_1 \cdot u_2\| < \varepsilon.$$

Since  $A$  has the strong strict comparison for positive elements,

$$\varphi_0(e) \lesssim \varphi_1(e). \quad (\text{e 4.17})$$

Put  $B_0 = \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$ . Let  $w_i = \lambda_i \cdot 1_{\tilde{B}_0} + y_i$ ,  $i = 0, 1, \dots, n$ . Then  $w_n = 1_{\tilde{B}_0}$ ,  $w_0 = 1 \cdot 1_{\tilde{B}_0} + y_0$  and

$$\|w_i - w_{i-1}\| < \varepsilon, \quad i = 1, 2, \dots$$

This implies that  $w_0 \in U_0(\tilde{B}_0)$ . Then by (e 4.17) and by 4.2,  $\text{cer}(u_1) \leq 2\pi$ . On the other hand, by 3.6,  $\text{cer}(u_2) \leq 2 + \varepsilon$ .

□

**Theorem 4.4.** *Let  $A$  be a separable simple  $C^*$ -algebra in  $\mathcal{D}$  and let  $u \in CU(\tilde{A})$ . Then  $u \in U_0(\tilde{A})$  and  $\text{cel}(u) \leq 5\pi$ .*

*Proof.* Let  $\pi : \tilde{A} \rightarrow \mathbb{C}$  be the quotient map. Since  $u \in CU(\tilde{A})$ ,  $\pi(u) = 1$ . So we write  $u = 1 + x_0$ , where  $x_0 \in A$ .

Let  $1/2 > \varepsilon > 0$ . There are  $v_1, v_2, \dots, v_k \in U(\tilde{A})$  such that

$$\|u - v_1 v_2 \cdots v_k\| < \varepsilon/16,$$

and  $v_i = a_i b_i a_i^* b_i$ ,  $a_i, b_i \in U(\tilde{A})$ . It is standard that in  $v_1 v_2 \cdots v_k \oplus 1_{M_{4k}} \in U_0(M_{4k+1}(\tilde{A}))$ . Since  $\tilde{A}$  has stable rank one (see 15.5 of [17]), by [42],  $v_1 v_2 \cdots v_k \in U_0(\tilde{A})$ . It follows that  $u \in U_0(\tilde{A})$ . Put  $u_0 = v_1 v_2 \cdots v_k$ . Let  $H = \text{cel}(u_0)$ .

Write  $a_i = \lambda_i + x_i$  and  $b_i = \mu_i + y_i$ , where  $|\lambda_i| = |\mu_i| = 1$  and  $x_i, y_i \in A$ ,  $i = 1, 2, \dots, k$ .

The rest of the proof is similar to that of 4.3. We will repeat some of the argument. we may assume that there is a strictly positive element  $e \in A_+$  such that  $\|e\| = 1$  and

$$f_\eta(e)x_i = x_i f_\eta(e) = x_i, f_\eta(e)y_i = y_i f_\eta(e) = y_i, \quad i = 0, 1, 2, \dots, k, \quad (\text{e 4.18})$$

for some  $\eta > 0$ . Let

$$\mathcal{G}_1 = \{e, f_\eta(e), f_{\eta/2}(e), x_i, y_i \mid 0 \leq i \leq k\}.$$

Put

$$d = \inf\{d_\tau(e) : \tau \in \overline{T(A)}^w\} > 0.$$

Without loss of generality, we may assume that  $\tau(f_{1/2}(e)) \geq d/2$  for all  $\tau \in \overline{T(A)}^w$ .

Choose  $n \geq 1$  such that

$$4H/n < \varepsilon/64k.$$

Let  $\delta > 0$  and let  $\mathcal{G} \supset A$  be a finite subset. Let  $e'_0 \in A_+ \setminus \{0\}$  such that

$$d_\tau(e'_0) < d/4(n+1) \quad \text{for all } \tau \in \overline{T(A)}^w.$$

Since  $A \in \mathcal{D}$ , there are  $A_0$  and  $M_n(D) \subset A$  with  $D \in \mathcal{C}'_0$ ,  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps  $\varphi_0 : A \rightarrow A_0$  and  $\varphi_1 : A \rightarrow M_n(D)$ , such that

$$\|x - \text{diag}(\varphi_0(x), \varphi_1(x))\| < \delta \quad \text{for all } x \in \mathcal{G} \quad (\text{e 4.19})$$

$$\varphi_0(e) \lesssim e'_0, \quad (\text{e 4.20})$$

$$\tau(f_{1/4}(\varphi_1(e))) \geq d/4n \quad \text{for all } \tau \in \overline{T(A)}^w. \quad (\text{e 4.21})$$

By choosing small  $\delta$  and large  $\mathcal{G}$ , we may assume the following: there is  $x'_0 \in \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$  such that  $1 + x'_0$  is a unitary,  $\text{cel}(p + x'_0) \leq 2H/n$ , where  $p$  is the unit of unitization of  $\tilde{B}$ , where  $B = \overline{\varphi_0(f_{\eta/2}(e))A\varphi_0(f_{\eta/2}(e))}$ , and there are  $z, z_i, x'_i, y'_i \in \overline{f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e))}$  such that  $\lambda_i + a'_i$  and  $\mu_i + b'_i$  are unitaries such that

$$\|(1+z) - (1+z_1)(1+z_2)\cdots(1+z_k)\| < \varepsilon/16 \quad \text{and} \quad \|u_0 - (1+x'_0 + z)\| < \varepsilon/16, \quad (\text{e 4.22})$$

where

$$1 + z_i = (\lambda_i \cdot 1 + x'_i)(\mu_i \cdot 1 + y'_i)(\lambda_i \cdot 1 + x'_i)^*(\mu_i + y'_i)^*, \quad i = 1, 2, \dots, k.$$

In particular,  $(1+z_1)(1+z_2)\cdots(1+z_k) \in CU(\tilde{C})$ , where  $C = \overline{f_{\eta/2}(\varphi_1(e))Df_{\eta/2}(\varphi_1(e))}$ . It follows from 3.6 that

$$\text{cel}((1+z_1)(1+z_2)\cdots(1+z_k)) \leq 4\pi.$$

As in the proof of 4.3, we have

$$\varphi_0(e) \lesssim \varphi_1(e).$$

As in the proof of 4.3, by applying 4.2, we have

$$\text{cel}(1 + x'_0) \leq 2H/n + 4\pi + \varepsilon < 4\pi + 2\varepsilon.$$

It follows that

$$\text{cel}(u) < 5\pi.$$

□

**Proposition 4.5.** (cf. Theorem 4.6 of [19]) *Let  $A$  be a separable simple  $C^*$ -algebra with continuous scale and let  $e \in A_+ \setminus \{0\}$ . Then the map  $\iota_e : U_0(\widetilde{e\bar{A}e})/CU(\widetilde{e\bar{A}e}) \rightarrow U_0(\tilde{A})/CU(\tilde{A})$  is surjective. If, in addition,  $A$  has stable rank one, then the map is also injective.*

*Proof.* The proof is almost identical to that of the unital case (see Theorem 4.6 of [19]).

First, we claim that, for any  $h \in A_{s.a.}$ , there exists  $h' \in (\widetilde{e\bar{A}e})_{s.a.}$  such that  $\tau(h') = \tau(h)$  for all  $\tau \in T(A)$ .

Put  $A_0 = \overline{e\bar{A}e}$ . By Proposition 9.5, there are  $x_i, y_j \in A$  ( $1 \leq i \leq n$  and  $1 \leq j \leq m$ ), such that

$$\sum_{i=1}^n x_i^* a_0 x_i = h'_+ \quad \text{and} \quad \sum_{j=1}^m y_j^* a_0 y_j = h'_-. \quad (\text{e 4.23})$$

Then

$$h'_0 := \sum_{i=1}^n a_0^{1/2} x_i^* x_i a_0^{1/2} - \sum_{j=1}^m a_0^{1/2} y_j^* y_j a_0^{1/2} \in A_0. \quad (\text{e 4.24})$$

Moreover,  $\tau(h_0) = \tau(h)$  for all  $\tau \in T(A)$ . This proves the claim.

To show  $\iota_e$  is surjective, let  $u \in U_0(\tilde{A})$  with  $u = \prod_{j=1}^l \exp(i2\pi h_j)$  with  $h_j \in \tilde{A}_{s.a.}$ . Write  $h_j = \alpha_j \cdot 1_{\tilde{A}} + h'_j$ , where  $\alpha_j \in \mathbb{R}$  with  $|\alpha_j| = 1$  and  $h'_j \in A_{s.a.}$ . By the claim that there  $h'_{0,j} \in (A_0)_{s.a.}$  such that  $\tau(h'_{0,j}) = \tau(h'_j)$  for all  $\tau \in T(A)$ . Let  $h_{0,j} = \alpha_j \cdot 1_{\tilde{A}_0} + h'_{0,j}$ ,  $j = 1, 2, \dots, l$ . Put  $w = \prod_{j=1}^l \exp(ih_{0,j})$ . Then  $w \in U_0(\tilde{A}_0)$ . Put  $v = \prod_{j=1}^l \exp(i\tilde{h}_{0,j})$ , where  $\tilde{h}_{0,j} = \alpha_j \cdot 1_{\tilde{A}} + h'_{0,j}$ ,  $j = 1, 2, \dots, l$ . Then  $v \in U_0(\tilde{A})$ . Moreover,  $\iota_e(\bar{w}) = \bar{v}$ . Since

$$D_{\tilde{A}}(v)(\tau) = \sum_{j=1}^l \tau(h_{0,j}) = \sum_{j=1}^l \tau(h_j) = D_{\tilde{A}}(u)(\tau) \quad (\text{e 4.25})$$

for all  $\tau \in T(\tilde{A})$ ,  $\iota_e(\bar{w}) = \bar{u}$ . This proves that  $\iota_e$  is surjective.

To see it is injective, let  $e_A \in A$  be a strictly positive element of  $A$  with  $\|e_A\| = 1$ . Since  $A$  has continuous scale, by (the proof of) Proposition 9.5, there exists an integer  $K \geq 1$  such that

$$K\langle a_0 \rangle > \langle e_A \rangle \quad (\text{e 4.26})$$

(in Cuntz semi-group). Since  $A$  has stable rank one, without loss of generality, we may write  $A \subset M_K(A_0)$ . Put  $E_0 = 1_{\tilde{A}_0}$ . Let  $u \in \tilde{A}_0$  with  $u = \lambda \cdot E_0 + x$  for some  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$  and  $x \in (A_0)_{s.a.}$ . Write  $w = \lambda \cdot 1_{\tilde{A}} + x$ . Then  $\iota_e(\bar{u}) = \bar{w}$ . Suppose that  $w \in CU(\tilde{A})$ . Write  $E = 1_{M_K(\tilde{A}_0)}$ . Write  $w' = \lambda \cdot E + x$ . Then  $w' \in CU(M_K(\tilde{A}_0))$ . However, since  $\tilde{A}_0$  has stable rank one, it follows from Theorem 4.6 that  $\bar{u} \in CU(\tilde{A}_0)$ . This shows that  $\iota_e$  is injective.

□

**Lemma 4.6.** *Let  $A$  be a non-unital and  $\sigma$ -unital simple  $C^*$ -algebra of stable rank one with continuous scale. Suppose that there is  $H > 0$  such that, for any hereditary  $C^*$ -subalgebra  $B$  of  $A$ ,  $\text{cel}(z) \leq H$  for any  $z \in CU(\tilde{B})$ . Suppose that there are two mutually orthogonal  $\sigma$ -unital*

hereditary  $C^*$ -subalgebras  $A_0$  and  $A_1$  with strictly positive elements  $a_0$  and  $a_1$  with  $\|a_0\| = 1$  and  $\|a_1\| = 1$ , respectively. Suppose that  $x \in A_0$  and suppose that for some  $\lambda \in \mathbb{C}$  with  $|\lambda| = 1$ ,  $w = \lambda + x \in U_0(\tilde{A})$ . Suppose also that there is an integer  $K \geq 1$  such that

$$Kd_\tau(a_0) \geq 1 \text{ for all } \tau \in T(A). \quad (\text{e 4.27})$$

Let  $u = \lambda \cdot 1_{\tilde{A}_0} + x$ . Suppose that, for some  $\eta \in (0, 2]$ ,

$$\text{dist}(\bar{w}, \bar{1}) \leq \eta.$$

Then, if  $\eta < 2$ , one has

$$\text{cel}_{\tilde{A}_0}(u) < \left(\frac{K\pi}{2} + 1/16\right)\eta + H \text{ and } \text{dist}(\bar{u}, \bar{1}_{\tilde{A}_0}) < (K + 1/8)\eta,$$

and if  $\eta = 2$ , one has

$$\text{cel}_{\tilde{A}_0}(u) < \frac{K\pi}{2}\text{cel}(w) + 1/16 + H.$$

*Proof.* Let  $L = \text{cel}(w)$ . It follows from 11.3 of [17] that, without loss of generality, we may assume that  $A_{00} = \overline{e_0 A e_0}$  has continuous scale, where  $e_0 \leq a_0$  and  $x e_0 = e_0 x$ . Note since  $A$  is simple and has stable rank one,  $u \in U_0(\tilde{A}_0)$ .

First consider the case that  $\eta < 2$ . Let  $c \in CU(\tilde{A})$  such that

$$\|c - w\| \leq \eta.$$

Choose  $\frac{\eta}{32K(K+1)\pi} > \varepsilon > 0$  such that  $\varepsilon + \eta < 2$ . Choose  $h \in A_{s.a.}$  such that with  $\|h\| \leq 2 \arcsin(\frac{\varepsilon + \eta}{2})$  such that

$$w \exp(ih) = c. \quad (\text{e 4.28})$$

Thus

$$\overline{D_{\tilde{A}}}(w \exp(ih)) = \bar{0} \text{ (in } \text{Aff}(T(\tilde{A}))/\overline{\rho_{\tilde{A}}(K_0(\tilde{A}))}). \quad (\text{e 4.29})$$

It follows that

$$|\overline{D_{\tilde{A}}}(w)(\tau)| \leq 2 \arcsin\left(\frac{\varepsilon + \eta}{2}\right). \quad (\text{e 4.30})$$

Put  $h = \alpha \cdot 1_{\tilde{A}} + h_0$ , where  $\alpha \in \mathbb{R}$  with  $|\alpha| = 1$  and  $h_0 \in A_{s.a.}$ . As in the proof of surjectivity of  $\iota_e$  in 4.5, there is  $h'_0 \in (A_0)_{s.a.}$  such that  $\tau(h'_0) = \tau(h_0)$  for all  $\tau \in T(A)$ . Put  $h''_0 = \alpha \cdot 1_{\tilde{A}} + h'_0$ . Moreover,  $\tau(h''_0) = \tau(h)$  for all  $\tau \in T(\tilde{A})$ . Therefore

$$\overline{D_{\tilde{A}}}(w \exp(ih''))(\tau) = \bar{0}. \quad (\text{e 4.31})$$

It follows from 4.5 that

$$D_{\tilde{A}_0}(u \exp(ih_{00})) = \bar{0} \text{ (in } \text{Aff}(T(\tilde{A}_0))/\overline{\rho_{\tilde{A}_0}(K_0(\tilde{A}_0))}), \quad (\text{e 4.32})$$

where  $h_{00} = \alpha \cdot 1_{\tilde{A}_0} + h'_0$ . By (e 4.27) and (e 4.30), in  $\tilde{A}_0$ ,

$$|\overline{D_{\tilde{A}_0}}(u)| \leq K 2 \arcsin\left(\frac{\varepsilon + \eta}{2}\right). \quad (\text{e 4.33})$$

Thus there is  $v \in CU(\tilde{A}_0)$  and  $h_1 \in \tilde{A}_{s.a.}$  such that

$$u = v \exp(2\pi i h_1) \text{ and } \|h_1\| \leq K 2 \arcsin\left(\frac{\varepsilon + \eta}{2}\right). \quad (\text{e 4.34})$$

Therefore



$$\text{cel}(u) \leq H + K2 \arcsin\left(\frac{\varepsilon + \eta}{2}\right) \leq H + K(\varepsilon + \eta)\frac{\pi}{2} \quad (\text{e 4.35})$$

$$\leq H + \left(K\frac{\pi}{2} + \frac{1}{64(K+1)}\right)\eta. \quad (\text{e 4.36})$$

One can also compute that

$$\text{dist}(\bar{u}, \bar{1}_{\bar{A}_0}) \leq K(\varepsilon + \eta) \leq K\eta + \frac{\eta}{32(K+1)\pi}.$$

This proves the case that  $\eta < 2$ .

Now suppose that  $\eta = 2$ . Define  $R = [\text{cel}(w) + 1]$ . Note that  $\frac{\text{cel}(w)}{R} < 1$ . Put  $w' = \lambda \cdot 1_{M_{R+1}} + x$ . It follows from 4.2 that

$$\text{dist}(\overline{w'}, \overline{1_{M_{R+1}}}) < \frac{\text{cel}(w)}{R+1} \quad (\text{e 4.37})$$

Put  $K_1 = K(R+1)$ . To simplify notation, replacing  $A$  by  $M_{R+1}(A)$ , without loss of generality, we may now consider that

$$K_1 d_\tau(a_0) \geq 1 \text{ and } \text{dist}(\bar{w}, \bar{1}) < \frac{\text{cel}(w)}{R+1}. \quad (\text{e 4.38})$$

Then we can apply the case that  $\eta < 2$  with  $\eta = \frac{\text{cel}(w)}{R+1}$ .

□

## 5 A Uniqueness theorem for $C^*$ -algebras in $\mathcal{D}$

**Proposition 5.1.** *Let  $A$  be a separable amenable  $C^*$ -algebra. Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset. Then there exists  $\delta > 0$  and  $\mathcal{G} \subset A$  satisfy the following: Suppose that there are two orthogonal  $C^*$ -subalgebras  $A_0$  and  $A_1$  and two  $\mathcal{F}$ - $\varepsilon/2$ -multiplicative completely positive contractive linear maps  $\varphi_0 : A \rightarrow A_0$  and  $\varphi_1 : A \rightarrow A_1$  such that*

$$\|x - \text{diag}(\varphi_0(x), \varphi_1(x))\| < \varepsilon/2 \text{ for all } x \in \mathcal{F}$$

*and suppose that there is  $\psi : A \rightarrow B$  (for any  $C^*$ -algebra  $B$ ) which is a  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map. Then there exist a pair of mutually orthogonal  $C^*$ -subalgebras  $B_0$  and  $B_1$  of  $B$  and  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear maps  $\psi_0 : A_0 \rightarrow B_0 \subset B$  and  $\psi_1 : A_1 \rightarrow B_1 \subset B$  such that*

$$\|\psi_0(x) - \psi \circ \varphi_0(x)\| < \varepsilon \text{ and} \quad (\text{e 5.1})$$

$$\|\psi_1(x) - \psi \circ \varphi_1(x)\| < \varepsilon \text{ for all } x \in \mathcal{F}. \quad (\text{e 5.2})$$

*Proof.* Fix  $1/2 > \varepsilon > 0$  and a finite subset  $\mathcal{F} \subset A$ . Let  $\{B_n\}$  be any sequence of  $C^*$ -algebras and let  $\varphi_n : A \rightarrow B_n$  be any sequence of completely positive contractive linear maps such that

$$\lim_{n \rightarrow \infty} \|\varphi_n(a)\varphi_n(b) - \varphi_n(ab)\| = 0 \text{ for all } a, b \in A. \quad (\text{e 5.3})$$

Let  $B_\infty = \prod_{n=1}^\infty B_n$ ,  $B_q = B_\infty / \oplus_{n=1}^\infty B_n$  and  $\Pi : B_\infty \rightarrow B_q$  is the quotient map. Define  $\Phi : A \rightarrow B_\infty$  by  $\Phi(a) = \{\varphi_n(a)\}$  for all  $a \in A$ . Then  $\Pi \circ \Phi : A \rightarrow B_q$  is a homomorphism. Suppose  $A_0$  and  $A_1$  are in the statement of the Proposition. Let  $a_0 \in (A_0)_+$  with  $\|a_0\| = 1$  and  $a_1 \in (A_1)_+$  with  $\|a_1\| = 1$  be strictly positive elements of  $A_0$  and  $A_1$ , respectively. Then  $a_0 a_1 = a_1 a_0 = 0$ . Therefore there are  $b^{(0)}, b^{(1)} \in B_\infty$  such that  $b^{(0)} b^{(1)} = b^{(1)} b^{(0)} = 0$  such that

$\Pi(b^{(i)}) = a_i$ ,  $i = 0, 1$ . Write  $b^{(i)} = \{b_n^{(i)}\}$ . Let  $B_{n,i} = \overline{b_n^{(i)} B_n b_n^{(i)}}$ ,  $i = 0, 1$ . Then  $B_{n,0}$  and  $B_{n,1}$  are mutually orthogonal. Since  $A$  is amenable, there is a completely positive contractive linear map  $\Psi : A \rightarrow B_\infty$  such that  $\Psi = \Pi \circ \Phi$ . Define  $\psi'_n : A \rightarrow B_n$  by

$$\psi'_n(a) = b_n^{(i)} \varphi_n(a) b_n^{(i)} \text{ for all } a \in A. \quad (\text{e5.4})$$

Let  $\psi_0 = \psi'_n \circ \varphi_0$  and  $\psi_1 = \psi'_n \circ \varphi_1$ . If  $n$  is sufficiently large, then  $\psi_0$  and  $\psi_1$  can be  $\mathcal{F}$ - $\varepsilon$ -multiplicative. Moreover, if  $n$  sufficiently large,

$$\|\psi_0(a) - \varphi_n \circ \varphi_0(a)\| < \varepsilon \text{ for all } a \in \mathcal{F} \text{ and} \quad (\text{e5.5})$$

$$\|\psi_1(a) - \varphi_n \circ \varphi_1(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad (\text{e5.6})$$

If the proposition fails, then such  $\{\varphi_n\}$  could not exist for some choice of  $\{B_n\}$ ,  $\varepsilon$  and  $\mathcal{F}$ . This proves the proposition.  $\square$

**5.2.** Fix a map  $\mathbf{T}(n, k) : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ . Let  $A \in \mathcal{D}$ . Denote by  $\mathcal{D}_{\mathbf{T}(n,k)}$  the class of  $C^*$ -algebras in  $\mathcal{D} \cap \mathbf{C}_{(r_0, r_1, T, s, R)}$  with  $r_0 = 0$ ,  $r_1 = 0$ ,  $\mathbf{T} = \mathbf{T}(k, n)$ ,  $s = 1$  and  $R = 6$ , as defined in 7.7 of [17].

Note if  $A \in \mathcal{D}$ , then  $A$  has stable rank one (see 15.5 of [17]) (so  $r_0 = 0$  and  $r_1 = 0$  in 7.7 of [17]) and by [17],  $\text{cer}(\tilde{A}) \leq 5 + \varepsilon$  ( $R \leq 6$ ). If  $A$  is also  $\mathcal{Z}$ -stable, then  $K_0(\tilde{A})$  is weakly unperforated. Thus  $A \in \mathcal{D}_{\mathbf{T}(n,k)}$  for  $\mathbf{T}(n, k) = n$  for all  $(n, k) \in \mathbb{N} \times \mathbb{N}$  (see 5.5 below).

**Theorem 5.3.** Fix  $\mathbf{T}(n, k)$ . Let  $A$  be a non-unital separable simple  $C^*$ -algebra in  $\mathcal{D}^d$  with continuous scale which satisfies the UCT. Let  $T : A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$  be a map. For any  $\varepsilon > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exists  $\delta > 0$ ,  $\gamma > 0$ ,  $\eta > 0$ , a finite subset  $\mathcal{G} \subset A$ , a finite subset  $\mathcal{H}_1 \subset A_+ \setminus \{0\}$ , a finite subset  $\mathcal{P} \subset \underline{K}(A)$ , a finite subset  $\mathcal{U} = \{v_1, v_2, \dots, v_{m_0}\} \subset U(\tilde{A})$  such that  $\{[v_1], [v_2], \dots, [v_{m_0}]\} = \mathcal{P} \cap K_1(A)$ , and a finite subset  $\mathcal{H}_2 \subset A_{s.a.}$  satisfy the following: Suppose that  $\varphi_1, \varphi_2 : A \rightarrow B$  are two  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps which are  $T$ - $\mathcal{H}_1$ -full, where  $B \in \mathcal{D}_{\mathbf{T}(n,k)}$  with continuous scale such that

$$[\varphi_1]|_{\mathcal{P}} = [\varphi_2]|_{\mathcal{P}}, \quad (\text{e5.7})$$

$$|\tau \circ \varphi_1(h) - \tau \circ \varphi_2(h)| < \gamma \text{ for all } h \in \mathcal{H}_2 \text{ and } \tau \in T(B) \text{ and} \quad (\text{e5.8})$$

$$\text{dist}(\overline{[\varphi_1(v_i)]}, \overline{[\varphi_2(v_i)]}) < \eta \text{ for all } v_i \in \mathcal{U}. \quad (\text{e5.9})$$

Then there exists a unitary  $w \in \tilde{B}$  such that

$$\|\text{Ad } w \circ \varphi_1(a) - \varphi_2(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad (\text{e5.10})$$

*Proof.* Fix a finite subset  $\mathcal{F}$  and  $1/4 > \varepsilon > 0$ . As pointed out in 5.2,  $B \in \mathbf{C}_{(0,0,T(k,n),1,6)}$  for all  $B \in \mathcal{D}$ , where  $T(k, n) = n + 4$  for all  $(k, n)$ . Without loss of generality, we may assume that  $\mathcal{F} \subset A^1$ .

Since  $A$  has the continuous scale,  $T(A)$  is compact (see 9.3 of [17]). Fix a strictly positive element  $a_0 \in A_+$  with  $\|a\| = 1$ . We may assume, without loss of generality, that

$$a_0 y = y a_0 = y, \quad a_0 \geq y^* y \text{ and } a_0 \geq y y^* \text{ for all } y \in \mathcal{F} \text{ and} \quad (\text{e5.11})$$

$$\tau(f_{1/4}(a_0)) \geq 1 - \varepsilon/2^{11} \text{ for all } \tau \in T(A). \quad (\text{e5.12})$$

Let  $T_1 : A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$  with  $T_1(a) = (N(a), M(a))$  ( $a \in A_+ \setminus \{0\}$ ) be the map given by 12.6 and 14.9 of [17] (in place of  $T$ ) for  $d = 1 - \varepsilon/4$ . Suppose that  $T(a) = (N_T(a), M_T(a))$  for  $a \in A_+ \setminus \{0\}$ .

Define  $T_2, T_3 : A_+ \setminus \{0\} \rightarrow \mathbb{N} \times (\mathbb{R}_+ \setminus \{0\})$  by  $T_2(a) = (N(a), (4/3)M(a))$  and  $T_3(a) = (N_T(a)N(a), (8/6)(M_T + 1)M(a))$  for all  $a \in A_+ \setminus \{0\}$ . Define  $\mathbf{L}(u) = 8\pi$  for all  $u \in U(\tilde{A})$ .

Let  $\delta_1 > 0$  (in place of  $\delta$ ), let  $\mathcal{G}_1 \subset A$  (in place of  $\mathcal{G}$ ) be a finite subset, let  $\mathcal{H}_{1,0} \subset A_+ \setminus \{0\}$  (in place of  $\mathcal{H}$ ) be a finite subset,  $\mathcal{P}_1 \subset \underline{K}(A)$  (in place of  $\mathcal{P}$ ) be a finite subset, let  $\mathcal{U}_1 \subset U(\tilde{A})$  (in place of  $\mathcal{U}$ ) be a finite subset and let  $K_1 \geq 1$  (in place of  $K$ ) be an integer given by 7.9 (together with 7.14) of [17] for the above  $T_3$  (in place of  $F$ ),  $\varepsilon/16$  (in place of  $\varepsilon$ ) and  $\mathcal{F}$ . We assume that  $a_0, f_{1/16}(a_0), f_{1/8}(a_0)$  and  $f_{1/4}(a_0) \in \mathcal{F} \cup \mathcal{H}_{1,0}$  (with  $r_0 = r_1 = 0$ ,  $T = T(k, n)$  above,  $s = 1$  and  $R = 6$ ).

We may also assume that,  $\delta_1$  is sufficiently small and  $\mathcal{G}_1$  is sufficiently large that  $[L_i]_{\mathcal{P}}$  is well-defined, and

$$[L_1]_{\mathcal{P}} = [L_2]_{\mathcal{P}},$$

provided that  $L_i$  is  $\mathcal{G}_1$ - $2\delta_1$ -multiplicative and

$$\|L_1(x) - L_2(x)\| < \delta_1 \text{ for all } x \in \mathcal{G}_1.$$

Without loss of generality, we may also assume that

$$\mathcal{F} \cup \mathcal{H}_{1,0} \cup \{xy : x, y \in \mathcal{F}\} \subset \mathcal{G}_1 \subset A^1.$$

Choose  $b_0 \in A_+ \setminus \{0\}$  with  $d_\tau(b_0) < 1/32(2K_1 + 1)$  for all  $\tau \in T(A)$ .

We choose a larger finite subset  $\mathcal{G}'_1$  of  $A$  and a smaller  $\delta'_1$  so that

$$\|[L(u)] - L(u)\| < \min\{1/4, \varepsilon \cdot \delta_1/2^{10}\}/8\pi \quad (\text{e 5.13})$$

for all  $u \in \mathcal{U}_1$  and

$$\|f_{1/8}(L(a_0)) - L(f_{1/8}(a_0))\| < 1/2^{10}(K_1 + 1) \quad (\text{e 5.14})$$

provided that  $L$  is a  $\mathcal{G}'_1$ - $\delta'_1$ -multiplicative completely positive contractive linear map (to any other  $C^*$ -algebra).

We may assume that  $0 < \delta'_1 \leq \frac{\varepsilon \cdot \delta_1}{2^{12}(K_1+1)}$ .

For each  $v \in \mathcal{U}_1$ , there is  $\alpha(v) \in \mathbb{C}$  and  $a(v) \in A$  such that

$$v = \alpha(v) \cdot 1_{\tilde{A}} + a(v), \quad |\alpha(v)| = 1 \text{ and } \|a(v)\| \leq 2. \quad (\text{e 5.15})$$

Let  $\Omega = \{a(v) : v \in \mathcal{U}_1\}$ . We may also assume that  $\mathcal{G}'_1 \supset \mathcal{G}_1 \cup \mathcal{F} \cup \mathcal{H}_{1,0} = \{xy : x, y \in \mathcal{G}_1\} \cup \Omega$ .

It follows from 12.6 and 14.9 of [17] that there are  $\mathcal{G}'_1$ - $\delta'_1/64$ -multiplicative completely positive contractive linear maps  $\varphi_0 : A \rightarrow A$  and  $\psi_0 : A \rightarrow D$  for some  $D \subset A$  with  $D \in \mathcal{C}'_0$  such that

$$\|x - \text{diag}(\varphi_0(x), \overbrace{\psi_0(x), \psi_0(x), \dots, \psi_0(x)}^{2K_1+1})\| < \min\{\varepsilon/K_1 2^{12}, \delta'_1/128K_1\} \text{ for all } x \in \mathcal{G}'_1, \quad (\text{e 5.16})$$

$$a'_{00} \lesssim b_0 \text{ and } \tau(f_{1/4}(\psi_0(a_0))) \geq 1 - \varepsilon/64 \text{ for all } \tau \in T(D) \quad (\text{e 5.17})$$

and  $\psi_0(a_0)$  is strictly positive, where  $a'_{00}$  is a strictly positive element of  $\overline{\varphi_0(a_0)A\varphi_0(a_0)}$ . Moreover,  $\psi_0$  is  $T_1$ - $\mathcal{H}_{1,0}$ -full in  $\overline{DAD}$ .

We compute that, by (e 5.12), (e 5.16) and (e 5.14),

$$2\tau(f_{1/8}(\psi_0(a_0))) \geq 3/4K_1 \text{ for all } \tau \in T(A). \quad (\text{e 5.18})$$

We also compute that (see (e 5.12) and (e 5.16)), for all  $\tau \in T(A)$ ,

$$\tau(\text{diag}(f_{1/4}(\varphi_0(a_0)), \overbrace{f_{1/4}(\psi_0(a_0)), f_{1/4}(\psi_0(a_0)), \dots, f_{1/4}(\psi_0(a_0))}^{2K_1+1})) > 1 - \varepsilon/2^9. \quad (\text{e 5.19})$$

Let  $A_{00} = \overline{\text{diag}(\varphi_0(a_0), \psi_0(a_0)) \text{Adiag}(\varphi_0(a_0), \psi_0(a_0))}$  and let  $\varphi_{00} : A \rightarrow A_{00}$  be defined by

$$\varphi_{00}(x) = \text{diag}(\varphi_0(x), \psi_0(x)) \text{ for all } x \in A.$$

Let  $a_{00} = \text{diag}(a'_{00}, \psi_0(a_0)) \in A_{00}$  be a strictly positive element of  $A_{00}$ .

By choosing even possibly smaller  $\delta'_1$  and larger  $\mathcal{G}'_1$ , if necessary, we may assume that  $[\varphi_{00}]|_{\mathcal{P}_1}$  is well defined and denote  $\mathcal{P}_2 = [\varphi_{00}](\mathcal{P}_1)$ . Moreover, we may also assume, without loss of generality, that

$$[L']|_{\mathcal{P}_2} = [L'']|_{\mathcal{P}_2}, \quad (\text{e 5.20})$$

if

$$\|L'(x) - L''(x)\| < \delta'_1 \text{ for all } x \in \mathcal{G}'_1$$

and  $L'$  and  $L''$  are  $\mathcal{G}'_1$ - $\delta'_1$ -multiplicative completely positive contractive linear maps.

We may also assume that

$$\|f_{\delta'}(a_{00})\varphi_{00}(x) - \varphi_{00}(x)\| < \delta'_1/2^{10} \text{ and} \quad (\text{e 5.21})$$

$$\|f_{\delta'}(a_{00})\varphi_{00}(x)f_{\delta'}(a_{00}) - \varphi_{00}(x)\| < \delta'_1/2^{10} \text{ for all } x \in \mathcal{G}'_1 \quad (\text{e 5.22})$$

for some  $1/64 > \delta' > 0$ . Furthermore,

$$\|f_{\delta'}(\psi_0(a_0))\psi_0(x) - \psi_0(x)\| < \delta'_1/2^{10} \text{ and} \quad (\text{e 5.23})$$

$$\|f_{\delta'}(\psi_0(a_0))\psi_0(x)f_{\delta'}(\psi_0(a_0)) - \psi_0(x)\| < \delta'_1/2^{10} \text{ for all } x \in \mathcal{G}'_1. \quad (\text{e 5.24})$$

It follows from (e 5.18) that  $a'_{00} \lesssim b_0 \lesssim f_{1/8}(\psi_0(a_0))$  and, by 3.1 of [17], there exists  $x_0 \in A$ .

$$f_{\delta'/256}(a'_{00})(x_0^* f_{1/8}(\psi_0(a_0))x_0) = f_{\delta'/256}(a'_{00}). \quad (\text{e 5.25})$$

Let  $g \in C_0((0, 1]_+)$  be such that  $\|g\| = 1$ ,  $g(t) = 0$  for all  $t \in (0, \delta'/64)$  and  $t \in (\delta'/8, 1]$ .

Put (keep in mind that  $A$  is projectionless and simple)

$$\sigma_0 = \inf\{\tau(g(a_{00})) : \tau \in T(A)\} > 0. \quad (\text{e 5.26})$$

Let  $\bar{D} = M_{2K_1}(D)$ . Let  $j_1 : D \rightarrow M_{2K_1}(D) = \bar{D}$  be defined by

$$j_1(d) = \text{diag}(d, d, \dots, d) \text{ for all } d \in D.$$

Let  $\iota_1 : \bar{D} \rightarrow A$  be the embedding. Let  $\varepsilon_1 = \min\{\varepsilon/2^{10}, \delta_1/2^{10}, \delta'_1/2^{10}\}$ . Choose a finite subset  $\mathcal{G}'_2 \subset \bar{D}$  which contains  $\bigoplus_{i=1}^{2K_1} \pi_i \circ \psi_0(\mathcal{G}'_1)$ , where  $\pi_i : \bigoplus_{i=1}^{2K_1} D \rightarrow D$  is the projection to the  $i$ -th summand.

Let  $e_d \in D_+$  with  $\|e_d\| = 1$  such that

$$\|f_{1/4}(e_d)y - y\| < \varepsilon_1/16 \text{ and } \|yf_{1/4}(e_d) - y\| < \varepsilon_1/16 \quad (\text{e 5.27})$$

for all  $y \in \psi_0(\mathcal{G}'_1)$ . Let  $\bar{e}_d = \text{diag}(\overbrace{e_d, e_d, \dots, e_d}^{2K_1})$ . Without loss of generality, we may assume that  $\bar{e}_d, f_{1/4}(\bar{e}_d) \in \mathcal{G}'_2$ .

Define  $\Delta : D_+^{q,1} \setminus \{0\} \rightarrow (0, 1)$  by, for  $d \in D_+^1 \setminus \{0\}$ ,

$$\Delta(\hat{d}) = \min\{\inf\{\tau \circ \iota_1(d) : \tau \in T(A)\}, \max\{\frac{1}{2^{10}M(d)^2N(d)} : d \in \hat{d}\}\}. \quad (\text{e 5.28})$$

For  $\varepsilon_1$ , choose  $\varepsilon_2 > 0$  (in place of  $\sigma$ ) associated with  $\varepsilon_1/16$  (in place of  $\varepsilon$ ) and  $1/16$  (in place of  $\sigma$ ) required by Lemma 3.3 of [17]. Without loss of generality, we may assume that  $\varepsilon_2 < \varepsilon_1$ .

Let  $\mathcal{G}_d$  (in place of  $\mathcal{G}$ ) be a finite subset,  $\mathcal{P}_d \subset K_0(\bar{D})$  (in place of  $\mathcal{P}$ ) be a finite subset,  $\mathcal{H}_{1,d} \subset (\bar{D})_+^1 \setminus \{0\}$  (in place of  $\mathcal{H}_1$ ) be a finite subset,  $\mathcal{H}_{2,d} \subset (\bar{D})_{s.a.}$  (in place of  $\mathcal{H}_2$ ) be a finite subset,  $\delta_d > 0$  (in place of  $\delta$ ),  $\gamma_d > 0$  (in place of  $\gamma$ ) required by Theorem 11.5 in [17] for  $C = \bar{D}$ ,  $\varepsilon_2/4$  (in place of  $\varepsilon$ ),  $\mathcal{G}'_2$  (in place of  $\mathcal{G}$ ) and  $\Delta$  above.

By (e 5.13), there is a finite subset  $\mathcal{U}_2 \subset U(\tilde{A}_{00})$  such that

$$\|\varphi_{00}(w) - w'\| < \min\{1/4, \varepsilon_1/2^{10}\}/8\pi \text{ for all } w \in \mathcal{U}_1 \quad (\text{e 5.29})$$

and for some  $w' \in \mathcal{U}_2$ . For each  $w' \in \mathcal{U}_2$ , there is  $\alpha(w') \in \mathbb{C}$  with  $|\alpha(w')| = 1$  and  $a(w') \in A_{00}$  with  $\|a(w')\| \leq 2$  such that

$$w' = \alpha(w') \cdot 1_{\tilde{A}_{00}} + a(w').$$

Define

$$\Omega_0 = \{a(w') : w' \in \mathcal{U}_2\}.$$

Note that by viewing  $\tilde{A}_{00}$  as a  $C^*$ -subalgebra of  $\tilde{A}$ , we may also view  $\mathcal{U}_2$  as a subset of  $\tilde{A}$ .

Let

$$\mathcal{G}_2 = \{a_{00}, f_{\delta'/4}(a_{00}), g(a_{00}), x_0, x_0^*\} \cup \mathcal{G}'_1 \cup \varphi_0(\mathcal{G}'_1) \cup \psi_0(\mathcal{G}'_1) \cup \mathcal{G}'_2 \cup \mathcal{G}_d \cup \mathcal{H}_{1,d} \cup \mathcal{H}_{2,d} \cup \Omega_0,$$

$$\mathcal{H}_1 = \{a_{00}, f_{\delta'/4}(a_{00}), f_{1/4}(a_0), f_{1/4}(\psi_0(a_0)), g(a_{00})\} \cup \mathcal{H}_{1,0} \cup \psi_0(\mathcal{H}_{1,0}) \cup \mathcal{H}_{1,d},$$

$$\mathcal{H}_2 = \mathcal{H}_1 \cup \mathcal{H}_{2,d}, \quad K_2 = 2^8 \max\{M(a)^2 N(a)^2 : a \in \mathcal{H}_1\},$$

$$\sigma_{00} = \frac{\sigma_0}{K_2}, \quad \delta_2 = \frac{\min\{\delta_1/16, \delta_d/4, \gamma/2, \eta/2, \delta'/256, \sigma_{00}/4\}}{4(K_1 + 1)},$$

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup (j_1)_*(\mathcal{P}_d) \cup \{[w'] : w' \in \mathcal{U}_2\},$$

$$\gamma = \frac{\gamma_d \cdot \delta' \cdot \sigma_{00}}{128(K_1 + 1)},$$

$\eta = 1/2^{10}(K_1 + 1)K_2$ , and  $\mathcal{U} = \mathcal{U}_1 \cup \mathcal{U}_3$

Now let  $\mathcal{G}_0$  (in place of  $\mathcal{G}$  and  $\delta_0$  (in place of  $\delta$ ) be as required by 5.1 for  $\mathcal{G}_2$  and  $\delta_2$ . Since  $\bar{D}$  is weakly semi-projective, we may choose even large  $\mathcal{G}_0$  and smaller  $\delta_0$  such that there is a homomorphism  $\Phi$  from  $\bar{D}$  such that

$$\|L(x) - \Phi(x)\| < \delta_2/2 \text{ for all } x \in \mathcal{G}_2 \cap \bar{D}$$

for any  $\mathcal{G}_0$ - $\delta_0$ -multiplicative completely positive contractive linear map  $L$  from  $\bar{D}$ . We also assume that

$$\|L(f_{\delta'/4}(a_{00})) - f_{\delta'/4}(L(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \quad (\text{e 5.30})$$

$$\|L(g(a_{00})) - g(L(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \quad (\text{e 5.31})$$

$$\tau(g(L(a_{00}))) > (1/2)\sigma_{00} \text{ for all } \tau \in T(C) \text{ and} \quad (\text{e 5.32})$$

$$\tau(f_{\delta'/128}(L(a'_{00}))) < 1/16(2K_1 + 1) \text{ for all } \tau \in T(C) \quad (\text{e 5.33})$$

$$\tau(f_{1/8}(L(\psi_0(a_0)))) \geq 1/K_2 \text{ for all } \tau \in T(C) \text{ (since } f_{1/4}(\psi_0(a_0)) \in \mathcal{H}_1) \quad (\text{e 5.34})$$

for any  $\mathcal{G}_0$ - $\delta_0$ -multiplicative completely positive contractive linear map  $L$  from  $A$  to  $C$  which is also  $T$ - $\mathcal{H}_1$ -full (used for (e 5.32) and (e 5.34)), where  $C$  is any  $C^*$ -algebra with  $T(C) \neq \emptyset$ .

Let  $\mathcal{G} = \mathcal{G}_2 \cup \mathcal{G}_0$  and  $\delta = \min\{\delta_0/2, \delta_2/2\}$ .

Now suppose that  $\varphi_1, \varphi_2 : A \rightarrow B$  satisfy the assumption of the theorem for the above chosen  $\mathcal{G}$ ,  $\delta$ ,  $\gamma$ ,  $\mathcal{P}$ ,  $\eta$ ,  $\mathcal{H}_1$ ,  $\mathcal{H}_2$  and  $\mathcal{U}$  (for  $T$ ).

Let  $\varphi_{i,0} = \varphi_i \circ \varphi_{00}$ ,  $i = 1, 2$ . Let  $\psi'_{i,1} : \bar{D} \rightarrow B$  be defined by  $(\varphi_i)|_{\bar{D}}$ . By applying 5.1, without loss of generality, we may assume that there are two pairs of hereditary  $C^*$ -subalgebras

$B_0, B_1$  and  $B'_0$  and  $B'_1$ , with  $B_0 \perp B_1$  and  $B'_0 \perp B'_1$  such that  $\varphi_1(A_{00}) \subset B_0$  and  $\psi_{1,1}(\bar{D}) \subset B_1$ ,  $\varphi_2(A_{00}) \subset B'_0$  and  $\psi_{2,1}(\bar{D}) \subset B'_1$ , and  $\varphi_i|_{A_{00}}$  is  $\mathcal{G}_2$ - $\delta_2$ -multiplicative,  $\psi_{1,1} : \bar{D} \rightarrow B_1$  and  $\psi_{2,1} : \bar{D} \rightarrow B'_1$  are homomorphisms such that

$$\|\psi'_{i,1}(x) - \psi_{i,1}(x)\| < \delta_2/2 \text{ for all } x \in \mathcal{G}_2 \cap \bar{D}, i = 1, 2. \quad (\text{e 5.35})$$

We may further assume, by (e 5.25) (and  $x_0 \in \mathcal{G}_2$ )

$$f_{\delta'/128}(\varphi_1(a'_{00})) \lesssim \psi_{1,1}(f_{1/16}(\psi_0(a_0))). \quad (\text{e 5.36})$$

Choose  $b_{00} \in B_+$  such that  $\tau(b_{00}) \geq 1/2$  for all  $\tau \in T(B)$ . Since both  $\varphi_1, \varphi_2$  are  $T$ - $\mathcal{H}_1$ -full,  $\psi_{1,0}$  and  $\psi_{2,0}$  are  $(4/3)T$ -( $\psi_0(\mathcal{H}_{1,0}) \cup \mathcal{H}_{1,d}$ )-full. We then compute that

$$\tau(\psi_{i,0}(x)) \geq \Delta(\hat{x}) \text{ for all } x \in \mathcal{H}_{1,d} \text{ and for all } \tau \in T(B). \quad (\text{e 5.37})$$

Then, by the choice of  $\mathcal{P}$ ,  $\mathcal{H}_{2,d}$  and  $\gamma$ , by applying 11.5 in [17], we obtain a unitary  $U'_1 \in \tilde{B}$  such that

$$\|\text{Ad } U'_1 \circ \psi_{2,1}(x) - \psi_{1,1}(x)\| < \varepsilon_2/4 \text{ for all } x \in \mathcal{G}'_2. \quad (\text{e 5.38})$$

In particular,

$$\|\text{Ad } U'_1 \circ \psi_{2,1}(\bar{e}_d) - \psi_{1,1}(\bar{e}_d)\| < \varepsilon_2/4. \quad (\text{e 5.39})$$

By applying Lemma 3.3 of [17], there is a unitary  $U''_1 \in \tilde{B}$  such that

$$\text{Ad } U''_1 \circ \text{Ad } U'_1 \circ \psi_{2,1}(x) \in \overline{\psi_{1,1}(\bar{e}_d)B\psi_{1,1}(\bar{e}_d)} \text{ for all } x \in \overline{\psi_{2,1}(\bar{e}_d)B\psi_{2,1}(\bar{e}_d)} \text{ and} \quad (\text{e 5.40})$$

$$\|(U''_1)^*cU''_1 - c\| < \varepsilon_1/16 \text{ for all } c \in \overline{\psi_{2,1}(\bar{e}_d)B\psi_{2,1}(\bar{e}_d)}. \quad (\text{e 5.41})$$

Put  $U_1 = U'_1U''_1$ . Then we have

$$\text{Ad } U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d)x f_{1/4}(\bar{e}_d)) \in \overline{\psi_{1,1}(\bar{e}_d)B\psi_{1,1}(\bar{e}_d)} \text{ for all } x \in A \text{ and} \quad (\text{e 5.42})$$

$$\|\text{Ad } U_1 \circ \psi_{2,1}(x) - \psi_{1,1}(x)\| < \varepsilon_1/4 \text{ for all } x \in j_1 \circ \psi_0(\mathcal{G}'_1). \quad (\text{e 5.43})$$

Let  $B' = \overline{(\text{Ad } U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d))B(\text{Ad } U_1 \circ \psi_{2,1}(f_{1/4}(\bar{e}_d)))}$  and let

$$B_p = \{b \in B : bx = xb = 0 \text{ for all } x \in B'\}. \quad (\text{e 5.44})$$

By the choice of  $\mathcal{H}_2$  and the assumption (e 5.8), for all  $\tau \in T(B)$ ,

$$|\tau(\varphi_{1,0}(f_{\delta'/4}(a_{00}))) - \tau(\varphi_{2,0}(f_{\delta'/4}(a_{00})))| < \min\{\gamma/2, \delta_2/2\}, \quad (\text{e 5.45})$$

With (e 5.30) in mind, by the assumption, we have that

$$\|f_{\delta'/4}(\varphi_i(a_{00})) - \varphi_i(f_{\delta'/4}(a_{00}))\| < \min\{\delta_2/2, \delta'/32\} \text{ and} \quad (\text{e 5.46})$$

$$\|g(\varphi_i(a_{00})) - \varphi_i(g(a_{00}))\| < \min\{\delta_2/2, \delta'/32\}, \quad (\text{e 5.47})$$

$i = 1, 2$ . We then compute that, by (e 5.30), by the choice of  $\mathcal{H}_2$  and  $\gamma$ , and by (e 5.32),

$$\tau(f_{\delta'/4}(\varphi_2(a_{00}))) \leq \min\{\delta_2/2, \delta'/32\} + \tau(\varphi_2(f_{\delta'/4}(a_{00}))) \quad (\text{e 5.48})$$

$$\leq \min\{\delta_2/2, \delta'/32\} + \gamma + \tau(\varphi_1(f_{\delta'/4}(a_{00}))) \quad (\text{e 5.49})$$

$$< \min\{\delta_2/2, \delta'/32\} + \gamma + \min\{\delta_2/2, \delta'/32\} \quad (\text{e 5.50})$$

$$+ \tau(f_{\delta'/4}(\varphi_1((a_{00})))) \quad (\text{e 5.51})$$

$$< \tau(g(\varphi_1(a_{00}))) + \tau(f_{\delta'/4}(\varphi_1((a_{00})))) \quad (\text{e 5.52})$$

$$\leq \tau(f_{\delta'/64}(\varphi_1(a_{00}))) \quad (\text{e 5.53})$$



for all  $\tau \in T(B)$ . It is important to note that

$$U_1^* f_{\delta'/2}(\varphi_2(a_{00}))U_1, f_{\delta'/64}(\varphi_1(a_{00})) \in B_p.$$

Also note that  $B_p$  is a hereditary  $C^*$ -subalgebra of  $B$ . Since  $B$  has strong strictly comparison for positive elements and  $B$  has stable rank one, by 3.2 of [17], there is a unitary  $U'_2 \in \tilde{B}_p$  such that

$$(U'_2)^* U_1^* f_{\delta'/2}(\varphi_2(a_{00}))U_1(U'_2) \in \overline{f_{\delta'/128}(\varphi_1(a_{00}))B f_{\delta'/128}(\varphi_1(a_{00}))} := B_{00}. \quad (\text{e 5.54})$$

Write  $U'_2 = \alpha \cdot 1_{\tilde{B}_p} + z$  with  $z \in B_p$  and  $\alpha \in \mathbb{C}$  with  $|\alpha| = 1$ . Put  $U_2 = \alpha \cdot 1_{\tilde{B}} + z$ . Then (e 5.54) still holds by replacing  $U'_2$  by  $U_2$ . Moreover,

$$U_2^* x U_2 = x \quad (\text{e 5.55})$$

for any  $x \in B'$ . In particular,

$$\|U_2^*(\text{Ad } U_1 \circ \psi_{2,1}(x))U_2 - \psi_{1,1}(x)\| < \varepsilon_1/4 + \varepsilon_1/16 = 5\varepsilon_1/16 \quad (\text{e 5.56})$$

for all  $x \in j_1 \circ \psi_0(\mathcal{G}'_1)$ .

Put  $\varphi'_{2,0} = \text{Ad } U_2 \circ \text{Ad } U_1 \circ \varphi_{2,0}$  and define  $\varphi''_{2,0} : A \rightarrow B_{00}$  by

$$\varphi''_{2,0}(x) = U_2^* U_1^* f_{\delta'/2}(\varphi_2(a_{00}))\varphi_{2,0}(x)f_{\delta'/2}(\varphi_2(a_{00}))U_1 U_2 \text{ for all } x \in A. \quad (\text{e 5.57})$$

By (e 5.21),  $\varphi''_{2,0}$  is  $\mathcal{G}'_1$ - $\delta'_1/2^4$ -multiplicative completely positive contractive linear map. Define  $\varphi'_{1,0} : A \rightarrow B_{00}$  by

$$\varphi'_{1,0}(x) = f_{\delta'/2}(\varphi_1(a_{00}))\varphi_{1,0}(x)f_{\delta'/2}(\varphi_1(a_{00})) \text{ for all } x \in A. \quad (\text{e 5.58})$$

which is also  $\mathcal{G}'_1$ - $\delta'_1/2^4$ -multiplicative completely positive contractive linear map. Now both  $\varphi'_{1,0}$  and  $\varphi''_{2,0}$  are completely positive contractive linear maps from  $A$  into  $B_{00}$ . Note that  $B$  is separable and simple and has stable rank one. From the assumption, (e 5.21) and (e 5.20), we have

$$[\varphi''_{2,0}]|_{\mathcal{P}} = [\varphi_{2,0}]|_{\mathcal{P}} = [\varphi_{1,0}]|_{\mathcal{P}} = [\varphi'_{1,0}]|_{\mathcal{P}}. \quad (\text{e 5.59})$$

It follows from the choice of  $\mathcal{U}_2$  and assumption (e 5.9) (as well as (e 5.21) and (e 5.45) among others) that

$$\text{dist}(\overline{[\varphi''_{2,0}(v)]}, \overline{[\varphi'_{1,0}(v)]}) < \eta + \delta'_1/2^4 \text{ for all } v \in \mathcal{U}_1 \quad (\text{e 5.60})$$

as elements in  $U(\tilde{B})/CU(\tilde{B})$ . It follows from (e 5.34) that

$$\tau(f_{\delta'/128}(\varphi_1(a_{00}))) > \tau(f_{\delta'/128}(\varphi_1(\psi_0(a_0)))) \geq 1/K_2 \text{ for all } \tau \in T(B). \quad (\text{e 5.61})$$

It follows from 4.6 that, in  $U(\tilde{B}_{00})$ , for all  $v \in \mathcal{U}_1$ ,

$$\text{cel}_{\tilde{B}_3}(\overline{[\varphi''_{2,0}(v)]} \overline{[\varphi'_{1,0}(v)]^*}) < (\frac{K_2\pi}{2} + \frac{1}{16})(\eta + \delta_2) + 6\pi \quad (\text{e 5.62})$$

$$\leq 7\pi < \mathbf{L}(v). \quad (\text{e 5.63})$$

Now let  $\tilde{\psi}_d = \psi_{1,1} \circ \text{diag}(\psi_0, \psi_0)$  and  $B_2 = \overline{\tilde{\psi}_d(A)B\tilde{\psi}_d(A)}$ . Let  $b'_{00} \in B_{00}$  be a strictly positive element with  $\|b'_{00}\| = 1$  and let  $b_2 \in B_2$  be a strictly positive element with  $\|b_2\| = 1$ . It follows from (e 5.36) that

$$b'_{00} \lesssim b_2. \quad (\text{e 5.64})$$

Recall that  $\psi_{1,1}$  and  $\psi_{2,1}$  are assumed to be homomorphisms which are  $T$ - $\psi_0(\mathcal{H}_{1,0})$ -full. Since  $\psi_0$  is  $T_1$ - $\mathcal{H}_{1,0}$ -full in  $D$ ,  $\tilde{\psi}_d$  is also  $T_3$ - $\mathcal{H}_{1,0}$ -full in  $B_2$ . Recall that

$$\psi_{1,1}(j_1 \circ \psi_0(x)) = \text{diag}(\overbrace{\tilde{\psi}_d(x), \tilde{\psi}_d(x), \dots, \tilde{\psi}_d(x)}^{K_1}) \text{ for all } x \in A. \quad (\text{e 5.65})$$

Now we are ready to apply the stable uniqueness theorem 7.9 in [17]. By that theorem, viewing  $B$  as a hereditary  $C^*$ -subalgebra of  $M_{K_1+1}(B_2)$ , there exists a unitary  $U_3 \in M_{K_1+1}(B_2)$  such that

$$\|U_3^* \text{diag}(\varphi_{2,0}''(x), \psi_{1,1}(j_1 \circ \psi_0(x)))U_3 - \text{diag}(\varphi_{1,0}'(x), \psi_{1,1}(j_1 \circ \psi_0(x)))\| < \varepsilon/16 \quad (\text{e 5.66})$$

for all  $x \in \mathcal{F}$ . It follows from (e 5.27), (e 5.58), (e 5.21) and (e 5.57) that

$$\|U_3^* \text{diag}(\varphi_{2,0}'(x), \psi_{1,1}(j_1 \circ \psi_0(x)))U_3 - \text{diag}(\varphi_{1,0}(x), \psi_{1,1}(j_1 \circ \psi_0(x)))\| \quad (\text{e 5.67})$$

$$< \varepsilon/16 + \varepsilon_1/4 + \delta_1/2^8 < \varepsilon/8 \quad (\text{e 5.68})$$

for all  $x \in \mathcal{F}$ . Since  $B$  has stable rank one, one easily find a unitary  $U_3' \in \tilde{B}$  such that the above hold with  $\varepsilon/7$  instead of  $\varepsilon/8$ .

Put  $U_4 = U_1 U_2 U_3'$ . It follows from (e 5.16), (e 5.56), (e 5.35) and above that, for all  $x \in \mathcal{F}$ ,

$$\begin{aligned} & \|\text{Ad } U_4 \circ \varphi_2(x) - \varphi_1(x)\| \\ & \leq \|U_4^* \varphi_2(\text{diag}(\varphi_{00}(x), j_1 \circ \psi_0(x)))U_4 - \varphi_1(\text{diag}(\varphi_{00}(x), j_1 \circ \psi_0(x)))\| + 2 \min\{\varepsilon/128, \delta_1'/128\} \\ & < \|(U_3')^* (\text{diag}(\varphi_{2,0}'(x), \psi_{1,1}(j_1 \circ \psi_0(x))))U_3' - \text{diag}(\varphi_{1,0}(x), \psi_{1,1}(j_1 \circ \psi_0(x)))\| \\ & \quad + 5\varepsilon_1/16 + \delta_2/2 + \varepsilon/64 \\ & < \varepsilon/7 + 5\varepsilon_1/16 + \delta_2/2 + \varepsilon/64 < \varepsilon \end{aligned} \quad (\text{e 5.69})$$

□

**Remark 5.4.** It is easy to see that, with (e 5.8), we may assume that  $[v_i] \neq \{0\}$  (see (e 2.9)).

**Theorem 5.5.** *Let  $A$  be a non-unital separable stably projectionless exact simple  $C^*$ -algebra with continuous scale which is  $\mathcal{Z}$ -stable and  $T(A) \neq \emptyset$ . Then  $K_0(\tilde{A})$  is weakly unperforated, i.e., if  $x \in K_0(\tilde{A})$  with  $kx \in K_0(\tilde{A})_+ \setminus \{0\}$  for some integer  $k \geq 1$ , then  $x \in K_0(\tilde{A})_+$ . Furthermore, if  $p, q \in M_s(\tilde{A})$  (for some  $s \geq 1$ ) are two projections such that  $\tau(q) < \tau(p)$  for all  $\tau \in T(\tilde{A})$ , then  $q \lesssim p$ .*

*Proof.* Put  $A_1 = \widetilde{A \otimes \mathcal{Z}}$ . Note that, since  $A$  is  $\mathcal{Z}$ -stable,  $A_1 = \tilde{A}$ . Let  $B = \tilde{A} \otimes \mathcal{Z}$  and let  $\iota : A_1 \rightarrow B$  be the embedding. Then  $\iota_{*0} : K_0(A_1) \rightarrow K_0(B)$  is an isomorphism. Let  $\pi_A : A_1 \rightarrow \mathbb{C}$  and  $\pi_{\mathcal{Z}} : B \rightarrow \mathcal{Z}$  be the quotient maps. Note that  $\pi_{\mathcal{Z}} \circ \iota = \pi_A$ . Let  $t_0$  be the tracial state of  $A_1$  which vanishes on  $A \otimes \mathcal{Z} = A$  and  $t_{\mathcal{Z}}$  be the tracial state of  $\mathcal{Z}$ . Note  $T(A_1) = T(A) \cup \{t_0\}$  and  $T(B) = T(B) \cup \{t_{\mathcal{Z}} \circ \pi_{\mathcal{Z}}\}$ .

Let  $x \in K_0(A_1)$  such that  $kx > 0$  in  $K_0(A_1)$  for some integer  $k \geq 1$ . Suppose that  $p, q \in M_s(A_1)$  are two projections such that  $[p] - [q] = x$  in  $K_0(A_1)$ . Then  $\tau(p) > \tau(q)$  for all  $\tau \in T(A_1)$ . It follows that  $\tau(\iota(p)) > \tau(\iota(q))$  for all  $\tau \in T(B)$ . Also  $pM_s(A \otimes \mathcal{Z})p \neq \{0\}$ . Note  $p \notin M_s(A \otimes \mathcal{Z})$  since  $A$  is stably projectionless. Therefore the ideal generated by  $p$  in  $M_s(B)$  contains  $q$ . Since  $B$  is  $\mathcal{Z}$ -stable, by 4.10 of [47],  $q \lesssim p$  in  $M_s(B)$ . Therefore there is a projection  $p_1 \leq p$  in  $M_s(B)$  such that  $[p_1] = x$ . There is a unitary  $w \in \mathcal{Z}$  such that  $w^* \pi_{\mathcal{Z}}(p_1)w = 1_{M_k}$ , where  $1_{M_k} \in M_s(\mathbb{C})$  is a scalar matrix of rank  $k \leq s$ . Since  $K_1(\mathcal{Z}) = \{0\}$ , there exists a unitary  $W \in M_s(B)$  such that  $\pi_{\mathcal{Z}}(W) = w$ . Then  $W^* p_1 W - 1_{M_k} \in \ker \pi_{\mathcal{Z}} = M_s(A \otimes \mathcal{Z})$ . Let  $e = W^* p_1 W$ . Then  $e \in M_s(A_1)$ . We compute that  $[e] = x$  in  $K_0(A_1)$ . This implies that  $x > 0$  and  $K_0(A_1)$  is weakly unperforated. □

**Remark 5.6.** It should be noted that, under the assumption of 5.5,  $\tau(q) < \tau(p)$  for all  $\tau \in T(A)$  does not imply  $q \lesssim p$ .

**Remark 5.7.** In Theorem 5.3, if both  $\varphi_1$  and  $\varphi_2$  map strictly positive elements to strictly positive elements, then, by 9.6 of [17], therefore the fullness condition can be replaced by  $\tau(f_{1/2}\varphi_1(e)), \tau(f_{1/2}(\varphi_2(e))) \geq d$  for some given  $1 > d > 0$  and a strictly positive element  $e \in A$  for all  $\tau \in T(B)$ . If furthermore,  $\varphi_1$  and  $\varphi_2$  are assumed to be homomorphisms, then,  $\tau \circ \varphi_i$  are tracial states of  $T(A)$  for all  $\tau \in T(B)$ . Therefore, the fullness condition can be dropped.

## 6 Models and range of invariant

**Lemma 6.1.** *Let  $A$  be an AF algebra and  $\varphi_1, \varphi_2 : A \rightarrow Q$  be two unital homomorphisms with  $(\varphi_1)_{*0} = (\varphi_2)_{*0}$ . Let  $n$  be a positive integer. Define  $B_i$  ( $i = 1, 2$ ) to be the  $C^*$ -subalgebra of  $C([0, 1], Q \otimes M_{n+1}) \oplus A$  given by*

$$B_i = \{(f, a) \in C([0, 1], Q \otimes M_{n+1}) \oplus A : \begin{array}{l} f(0) = \varphi_i(a) \otimes \text{diag}(\overbrace{1, \dots, 1}^n, 0) \\ f(1) = \varphi_i(a) \otimes \text{diag}(\overbrace{1, \dots, 1}^{n+1}) \end{array}\} \quad (\text{e 6.1})$$

for  $i = 1, 2$ . Then  $B_1 \cong B_2$ .

*Proof.* Since both  $A$  and  $Q$  are AF algebras and  $(\varphi_1)_{*0} = (\varphi_2)_{*0}$ , there is a unitary path  $\{u(t)\}_{0 \leq t < 1}$  such that  $\varphi_2(a) = \lim_{t \rightarrow 1} u(t)\varphi_1(a)u(t)^*$  (see [30]). Define the isomorphism  $\psi : B_1 \rightarrow B_2$  by sending  $(f, a) \in B_1$  to  $(g, a) \in B_2$ , where  $g$  is given by

$$g(t) = \begin{cases} (u(|2t-1|) \otimes \mathbf{1}_{n+1})f(t)(u(|2t-1|) \otimes \mathbf{1}_{n+1})^* & \text{if } t \in (0, 1), \\ \varphi_2(a) \otimes \text{diag}(\overbrace{1, \dots, 1}^n, 0) & \text{if } t = 0, \\ \varphi_2(a) \otimes \text{diag}(\overbrace{1, \dots, 1}^{n+1}) & \text{if } t = 1. \end{cases}$$

It is straight forward to verify that  $g$  is continuous, that  $(g, a) \in B_2$ , and that  $\psi$  defines a desired isomorphism. □

**Definition 6.2.** Let  $G_0$  and  $G_1$  be two countable abelian groups. Let  $A$  be a unital AH-algebra with  $TR(A) = 0$ , unique tracial state,  $K_1(A) = G_1$  and  $K_0(A) = \mathbb{Q} \oplus G_0$  with  $\ker \rho_A = G_0$  and  $[1_A] = (1, 0)$ .

There is a unital homomorphism  $s : A \rightarrow Q$  such that  $s_{*0}(r, g) = r$  for  $(r, g) \in \mathbb{Q} \oplus G_0$ . Fix a unital embedding  $j : Q \rightarrow A$  with  $j_{*0}(r) = (r, 0)$  for  $r \in \mathbb{Q}$ . (Note that both  $j \circ s$  and  $s \circ j$  induce the identity maps on  $T(A)$  and  $T(Q)$  respectively. Furthermore the homomorphism  $j$  and  $s$  identify the spaces  $T(A)$  and  $T(Q)$ )

Fix an integer  $a_1 \geq 1$ . Let  $\alpha = \frac{a_1}{a_1+1}$ . For each  $r \in \mathbb{Q}_+ \setminus \{0\}$ , let  $e_r \in Q$  be a projection with  $\text{tr}(e_r) = r$ . Let  $\bar{Q}_r := (1 \otimes e_r)(Q \otimes Q)(1 \otimes e_r)$ . Define  $q_r : Q \rightarrow \bar{Q}_r$  by  $a \mapsto a \otimes e_r$  for  $a \in Q$ . We will also use  $q_r$  to denote a homomorphism from  $B$  to  $B \otimes e_r Q e_r$  (or to  $B \otimes Q$ ) defined by sending  $b \in B$  to  $b \otimes e_r \in B \otimes e_r Q e_r \subset B \otimes Q$ .

We fix an isomorphism  $Q \otimes Q \rightarrow Q$  which will be denoted by  $\iota^Q$ . Moreover the composition of the maps which first maps  $a$  to  $a \otimes 1_Q$  and then to  $Q$  via  $\iota^Q$  is approximately inner. In fact every unital endomorphism on  $Q$  is approximately inner. If we identify  $Q$  with  $Q \otimes 1_Q$  in  $Q \otimes Q$  then  $\iota^Q$  is an approximately inner endomorphism.

For each  $1 > r > r' > 0$ , we assume that  $e_r \geq e_{r'}$ . Fix  $1 > r > 0$ , define  $\iota_r^Q : \bar{Q}_r \rightarrow Q_r := e_r Q e_r$  by  $\iota_r^Q = \text{Ad } v_r \circ \iota^Q|_{\bar{Q}_r}$ , where  $v_r^*(\iota^Q(1 \otimes e_r))v_r = e_r$ .

Let

$$R(\alpha, r) = \{(f, a) \in C([0, 1], Q \otimes Q_r) \oplus Q : f(0) = a \otimes e_{r\alpha} \text{ and } f(1) = a \otimes e_r\}.$$

(Recall that  $R(\alpha, 1)$  has been defined in 3.10.)

Let

$$A(W, \alpha) = \{(f, a) \in C([0, 1], Q \otimes Q) \oplus A : f(0) = q_\alpha \circ s(a) \text{ and } f(1) = s(a) \otimes 1_Q\}.$$

We also note that  $(f, a)$  is full in  $A(W, \alpha)$  if and only if  $a \neq 0$  and  $f(t) \neq 0$  for all  $t \in (0, 1)$ .

Let  $\mathcal{M}_+$  denote the set of nonnegative regular measures on  $(0, 1)$ . As in 3.7, trace spaces  $\tilde{T}(A(W, \alpha))$  and  $\tilde{T}(R(\alpha, 1))$  are isomorphic, and each  $\tau \in \tilde{T}(R(\alpha, 1)) \cong \tilde{T}(A(W, \alpha))$  corresponds to  $(\mu, s) \in \mathcal{M}_+(0, 1) \times \mathbb{R}_+$ . Furthermore we have

$$\|\tau\| = \|\mu\| + s = \int_0^1 d\mu + s.$$

Note that in the weak topology of  $\tilde{T}(A(W, \alpha))$  (or  $\tilde{T}(R(\alpha, 1))$ ), under the above identification, one has that

$$\lim_{t \rightarrow 0} (\delta_t, 0) = (0, \alpha) \in \mathcal{M}_+(0, 1) \times \mathbb{R}_+ \quad \text{and} \quad \lim_{t \rightarrow 1} (\delta_t, 0) = (0, 1) \in \mathcal{M}_+(0, 1) \times \mathbb{R}_+,$$

where  $\delta_t$  is the unit measure of the point mass at  $t$ .

The affine space  $\text{Aff}(\tilde{T}(A(W, \alpha)))$  and  $\text{Aff}(\tilde{T}(R(\alpha, 1)))$  can be identified with

$$\{(f, x) \in C([0, 1], \mathbb{R}) \oplus \mathbb{R} : f(0) = \alpha \cdot x \text{ and } f(1) = x\}, \quad (\text{e6.2})$$

a subspace of  $C([0, 1], \mathbb{R}) \oplus \mathbb{R}$ .

Let

$$A(W, \alpha, r) = \{(f, a) \in C([0, 1], Q \otimes Q_r) \oplus A : f(0) = q_{r\alpha} \circ s(a) \text{ and } f(1) = q_r \circ s(a)\}.$$

Define  $\varphi_{A, R, \alpha} : A(W, \alpha) \rightarrow R(\alpha, 1)$  by

$$\varphi_{A, R, \alpha}((f, a)) = (f, s(a)) \text{ for all } (f, a) \in A(W, \alpha).$$

Define  $\tilde{s}j : C([0, 1], Q \otimes Q) \rightarrow C([0, 1], Q \otimes Q)$  by

$$\tilde{s}j(f)(t) = ((s \circ j) \otimes \text{id}_Q)(f(t)).$$

Define  $\varphi_{R, A, \alpha} : R(\alpha, 1) \rightarrow A(W, \alpha, 1)$  by

$$\varphi_{R, A, \alpha}((f, a)) = (\tilde{s}j(f), j(a)) \text{ for all } (f, a) \in R(\alpha, 1).$$

Note that

$$\tilde{s}j(f)(0) = ((s \circ j) \otimes \text{id}_Q)(a \otimes e_\alpha) = s \circ j(a) \otimes e_\alpha \text{ and} \quad (\text{e6.3})$$

$$\tilde{s}j(f)(1) = ((s \circ j) \otimes \text{id}_Q)(a \otimes 1) = s \circ j(a) \otimes 1. \quad (\text{e6.4})$$

Also

$$q_\alpha \circ s \circ j(a) = s \circ j(a) \otimes e_\alpha.$$

In particular,  $\varphi_{R,A,\alpha}$  does map  $R(\alpha, 1)$  into  $A(W, \alpha, 1)$ . Moreover  $\varphi_{R,A,\alpha}$  is injective and map the strictly positive element  $a_\alpha$  to a strictly positive element (with the same form—see 3.10).

With the identification of both  $\text{Aff}(\tilde{T}(A(W, \alpha)))$  and  $\text{Aff}(\tilde{T}(R(\alpha, 1)))$  with the same subspace of  $C([0, 1], \mathbb{R}) \oplus \mathbb{R}$ , the homomorphism  $\varphi_{A,R,\alpha}$  and  $\varphi_{R,A,\alpha}$  induce the identity map on that subspace at the level of  $\text{Aff}(\tilde{T}(-))$  maps. They also induce the identity maps at level of trace spaces, when we identify the corresponding trace spaces. In particular,  $\varphi_{A,R,\alpha}^* : \tilde{T}(R(\alpha, 1)) \rightarrow \tilde{T}(A(W, \alpha))$  (or  $\varphi_{R,A,\alpha}^* : \tilde{T}(A(W, \alpha)) \rightarrow \tilde{T}(R(\alpha, 1))$ , respectively) takes the subset  $T(R(\alpha, 1))$  to the subset  $T(A(W, \alpha))$  (or takes  $T(A(W, \alpha))$  to  $T(R(\alpha, 1))$ , respectively)

Fix  $\alpha, r \in \mathbb{Q}_+ \setminus \{0\}$ . There are unitaries  $u_{\alpha,r}, u_{1,r} \in \bar{Q}_r$  such that

$$u_{\alpha,r}^*(e_\alpha \otimes e_r)u_{\alpha,r} = (\iota_r^Q)^{-1}(e_{r\alpha}) \quad \text{and} \quad u_{1,r}^*(1 \otimes e_r)u_{1,r} = (\iota_r^Q)^{-1}(e_r) = 1 \otimes e_r.$$

(Note that  $u_{1,r}$  can be chosen to be  $1_{\bar{Q}_r}$ .)

There is a continuous path of unitaries  $\{u(t) : t \in [0, 1]\}$  in  $\bar{Q}_r$  such that  $u(0) = u_{\alpha,r}$  and  $u(1) = u_{1,r}$ .

Let  $v(t) = 1 \otimes u(t) \in Q \otimes \bar{Q}_r$  for  $t \in [0, 1]$ . Note if  $f(t) \in Q \otimes Q$ , then

$$v(t)^*(f(t) \otimes e_r)v(t) \in Q \otimes \bar{Q}_r \quad \text{for all } t \in (0, 1).$$

Let  $\varphi_{R,r} : R(\alpha, 1) \rightarrow R(\alpha, r)$  be defined by

$$\varphi_{R,r}((f, a)) = (\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v \circ q_r(f), a).$$

Note that, for  $t \in (0, 1)$ ,

$$(\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(t) \circ q_r(f)(t) = (\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(t)(f(t) \otimes e_r) \quad (\text{e6.5})$$

$$= (\text{id}_Q \otimes \iota_r^Q)(v(t)^* f(t) \otimes e_r)v(t) \in Q \otimes Q_r, \quad (\text{e6.6})$$

$$(\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(0) \circ q_r(f)(0) = (\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(0)(a \otimes e_\alpha \otimes e_r) \quad (\text{e6.7})$$

$$= (\text{id}_Q \otimes \iota_r^Q)(a \otimes (\iota_r^Q)^{-1}(e_{\alpha r})) \quad (\text{e6.8})$$

$$= a \otimes e_{\alpha r} \quad \text{and} \quad (\text{e6.9})$$

$$(\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(1) \circ q_r(f)(1) = (\text{id}_Q \otimes \iota_r^Q) \circ \text{Ad } v(1)(a \otimes 1 \otimes e_r) \quad (\text{e6.10})$$

$$= (\text{id}_Q \otimes \iota_r^Q)(a \otimes (\iota_r^Q)^{-1}(e_r)) \quad (\text{e6.11})$$

$$= a \otimes e_r. \quad (\text{e6.12})$$

Evidently, when we identify  $\tilde{T}(R(\alpha, r))$  and  $\tilde{T}(R(\alpha, 1))$  with  $\mathcal{M}_+(0, 1) \times \mathbb{R}_+$ , the map  $\varphi_{R,r}^*$  is the identity map and takes the subset  $T(R(\alpha, r))$  to the subset  $T(R(\alpha, 1))$ .

Define  $s^{(2,3)} : Q \otimes Q \otimes Q \rightarrow Q \otimes Q \otimes Q$  by

$$s^{(2,3)}(x \otimes y \otimes z) = (x \otimes z) \otimes y$$

for all  $x, y, z \in Q$ . Define a homomorphism  $\widetilde{\iota^Q} : R(\alpha, 1) \otimes Q \rightarrow R(\alpha, 1)$  by

$$\widetilde{\iota^Q}(f \otimes b, a \otimes b) = ((\iota^Q \otimes \text{id}_Q) \circ s^{(2,3)}(f \otimes b), \iota^Q(a \otimes b))$$

for  $(f, a) \in R(\alpha, 1)$  and  $b \in Q$ .

Note, at  $t = 0$ ,

$$(\iota^Q \otimes \text{id}_Q) \circ s^{(2,3)}(f \otimes b)(0) = (\iota^Q \otimes \text{id}_Q) \circ s^{(2,3)}(a \otimes e_\alpha \otimes b) \quad (\text{e6.13})$$

$$= (\iota^Q \otimes \text{id}_Q)((a \otimes b) \otimes e_\alpha) \quad (\text{e6.14})$$

$$= \iota^Q(a \otimes b) \otimes e_\alpha; \quad (\text{e6.15})$$

and, at  $t = 1$ ,

$$(\iota^Q \otimes \text{id}_Q) \circ s^{(2,3)}(f \otimes b)(1) = (\iota^Q \otimes \text{id}_Q) \circ s^{(2,3)}(a \otimes 1 \otimes b) \quad (\text{e 6.16})$$

$$= (\iota^Q \otimes \text{id}_Q)((a \otimes b) \otimes 1) \quad (\text{e 6.17})$$

$$= \iota^Q(a \otimes b) \otimes 1. \quad (\text{e 6.18})$$

Let  $m \geq 2$  be an integer. Viewing  $M_m$  as a unital  $C^*$ -subalgebra of  $Q$ , Put  $\iota^{M_m} = \iota^Q|_{Q \otimes M_m}$ . Define  $\widetilde{\iota^{M_m}} : R(\alpha, 1) \otimes M_m \rightarrow R(\alpha, 1)$  by  $\widetilde{\iota^{M_m}} = \widetilde{\iota^Q}|_{R(\alpha, 1) \otimes M_m}$ . Note also that (recall (e 3.19))

$$\widetilde{\iota^Q}(a_\alpha \otimes 1_Q) = a_\alpha \text{ and } \widetilde{\iota^{M_m}}(a_\alpha \otimes 1_{M_m}) = a_\alpha. \quad (\text{e 6.19})$$

We need one more map. Let  $\psi_{A_w} : A(W, \alpha) \rightarrow C([0, 1], Q) \oplus A$  be defined by

$$\psi_{A_w}(f, a) = (g, a),$$

where  $g(t) = s(a)$  for all  $t \in [0, 1]$ . Define  $\psi_{A_w, r} : A(W, \alpha) \rightarrow C([0, 1], Q \otimes Q_r) \oplus A$  by

$$\psi_{A_w, r}((f, a)) = (q_r(g), a)$$

with  $g(t) = s(a)$  (and  $q_r(g) = q_r \circ s(a)$ ). Note that  $\psi_{A_w, r}(a_\alpha) = (1 \otimes e_r, 1)$  is the unit of  $C([0, 1], Q \otimes Q_r) \oplus A$ . It follows that  $\psi_{A_w, r}$  maps strictly positive elements to strictly positive elements.

When we identify  $\tilde{T}(A(W, \alpha))$  with  $\mathcal{M}_+(0, 1) \times \mathbb{R}_+$ , and  $\tilde{T}(C([0, 1], Q \otimes Q_r) \oplus A)$  with  $\mathcal{M}_+[0, 1] \times \mathbb{R}_+$ , the map  $\psi_{A_w, r}^*$  is given by

$$\psi_{A_w, r}^*(\mu, s) = (0, s + \int_0^1 d\mu),$$

which takes  $T(C([0, 1], Q \otimes Q_r))$  to  $T(A(W, \alpha))$ .

Warning:  $C([0, 1], Q \otimes Q_r) \oplus A \neq A(W, \alpha)$ .

One more notation: define  $P_f : (f, a) \rightarrow f$  and  $P_a : (f, a) = a$ .

Now let  $\alpha < \beta < 1$ . Let us choose  $x$  such that  $\beta(1/2 + x) = (\alpha/2 + x)$ . So

$$x = \frac{(1/2)(\beta - \alpha)}{1 - \beta} > 0.$$

Let

$$y = 1/2 + x = \frac{1}{2} + \frac{(1/2)(\beta - \alpha)}{(1 - \beta)} = \frac{(1 - \alpha)}{2(1 - \beta)}.$$

Let  $r_1 = (1/2)(1/y) = \frac{(1-\beta)}{(1-\alpha)}$  and  $r_2 = x(1/y) = \frac{(\beta-\alpha)}{(1-\alpha)}$ . Then

$$\alpha r_1 + r_2 = (1/y)(1/2 + x) = \beta \text{ and } r_1 + r_2 = (1/y)(1/2 + x) = 1.$$

Define  $\Phi_{A_w, \alpha, \beta} : A(W, \alpha) \rightarrow A(W, \beta)$  by

$$P_a(\Phi_{A_w, \alpha, \beta}((f, a))) = a \text{ and}$$

$$P_f(\Phi_{A_w, \alpha, \beta}((f, a))) = \text{diag}(P_f \circ \varphi_{R, r_1} \circ \varphi_{A, R, \alpha}((f, a)), P_f \circ \psi_{A_w, r_2}((f, a))).$$

One computes that, for  $t \in (0, 1)$ ,

$$P_f(\varphi_{R, r_1} \circ \varphi_{A, R, \alpha}((f, a)))(t) = (\text{id}_Q \otimes \iota_{r_1}^Q) \circ \text{Ad } v(t) \circ q_{r_1}(f)(t) \quad (\text{e 6.20})$$

$$= (\text{id}_Q \otimes \iota_{r_1}^Q)(v(t)^* f(t) \otimes e_{r_1} v(t)) \quad (\text{e 6.21})$$

$$\in Q \otimes Q_{r_1} \subset Q \otimes Q \text{ and} \quad (\text{e 6.22})$$

$$P_f(\psi_{A_w, r_2}((f, a)))(t) = q_{r_2}(s(a)) = s(a) \otimes e_{r_2} \in Q \otimes Q. \quad (\text{e 6.23})$$



At  $t = 0$ ,

$$P_f(\varphi_{R,r_1} \circ \varphi_{A,R,\alpha}((f,a)))(0) = (\text{id}_Q \otimes \iota_{r_1}^Q) \circ \text{Ad } v(0) \circ q_{r_1}(f)(0) \quad (\text{e 6.24})$$

$$= (\text{id}_Q \otimes \iota_{r_1}^Q) \circ \text{Ad } v(0)(s(a) \otimes e_\alpha \otimes e_{r_1}) \quad (\text{e 6.25})$$

$$= (\text{id}_Q \otimes \iota_{r_1}^Q)(a \otimes (\iota_{r_1}^Q)^{-1}(e_{\alpha r_1})) \quad (\text{e 6.26})$$

$$= s(a) \otimes e_{\alpha r_1}. \quad (\text{e 6.27})$$

Hence

$$P_f(\Phi_{A_w,\alpha,\beta}((f,a)))(0) = \text{diag}(s(a) \otimes e_{\alpha r_1}, s(a) \otimes e_{r_2}) \quad (\text{e 6.28})$$

$$= s(a) \otimes e_{\alpha r_1 + r_2} = s(a) \otimes e_\beta. \quad (\text{e 6.29})$$

At  $t = 1$ ,

$$P_f(\varphi_{R,R,r_2} \circ \varphi_{A,R,\alpha}((f,a)))(1) = (\text{id}_Q \otimes \iota_{r_1}^Q) \circ \text{Ad } v(1) \circ q_{r_1}(f)(1) \quad (\text{e 6.30})$$

$$= (\text{id}_Q \otimes \iota_{r_1}^Q) \circ \text{Ad } v(1)(s(a) \otimes 1 \otimes e_{r_1}) \quad (\text{e 6.31})$$

$$= (\text{id}_Q \otimes \iota_{r_1}^Q)(s(a) \otimes (\iota_{r_1}^Q)^{-1}(e_{r_1})) \quad (\text{e 6.32})$$

$$= s(a) \otimes e_{r_1}. \quad (\text{e 6.33})$$

Hence

$$P_f(\Phi_{A_w,\alpha,\beta}((f,a)))(1) = \text{diag}(s(a) \otimes e_{r_1}, s(a) \otimes e_{r_2}) \quad (\text{e 6.34})$$

$$= s(a) \otimes e_{r_1 + r_2} = s(a) \otimes 1. \quad (\text{e 6.35})$$

Therefore, indeed,  $\Phi_{A_w,\alpha,\beta}$  defines a homomorphism from  $A(W, \alpha)$  to  $A(W, \beta)$ . It is injective. We also check that  $\Phi_{A_w,\alpha,\beta}(a_\alpha)$  is a strictly positive element of  $A(W, \beta)$  (recall (e 3.19)).

Furthermore  $\Phi_{A_w,\alpha,\beta}^* : \tilde{T}(A(W, \beta))(\cong \mathcal{M}_+(0, 1) \times \mathbb{R}_+) \rightarrow \tilde{T}(A(W, \alpha))(\cong \mathcal{M}_+(0, 1) \times \mathbb{R}_+)$  is given by

$$\Phi_{A_w,\alpha,\beta}^*(\mu, s) = (r_1\mu, r_2(\int_0^1 d\mu) + s),$$

which takes  $T(A(W, \beta))$  to  $T(A(W, \alpha))$ .

Fix any  $a \in A_+$  with  $\|a\| = 1$ . Define  $f(t) = (1-t)(s(a) \otimes e_\alpha) + t(s(a) \oplus 1)$ . Then  $(f, a) \in A(\alpha, 1)$  is a full positive element. Note that  $\Phi_{A_w,\alpha,\beta}((f, a))$  is also a full positive element.

Let  $m, m'$  be two positive integers such that  $m|m'$ . Let  $\frac{m'}{m} = a + 1$ . Let  $F_2 = M_{m'}(\mathbb{C})$ ,  $F_1 = M_m(\mathbb{C})$ , and  $\varphi_0, \varphi_1 : F_1 \rightarrow F_2$  be defined by

$$\varphi_0(x) = \text{diag}(\underbrace{x, \dots, x}_a, 0), \quad \text{and} \quad \varphi_1(x) = \text{diag}(\underbrace{x, \dots, x}_{a+1}).$$

Denote that

$$A(m, m') = A(F_1, F_2, \varphi_0, \varphi_1) = \{(f, x) \in C([0, 1], M_m(\mathbb{C}) \otimes M_{a+1}(\mathbb{C})) \oplus M_m(\mathbb{C}) :$$

$$f(0) = x \otimes \text{diag}(\underbrace{1, \dots, 1}_a, 0) \quad f(1) = x \otimes \text{diag}(\underbrace{1, \dots, 1}_{a+1})\}.$$

Then  $A(m, m') \in \mathcal{C}_0^0$  with  $\lambda_s(A(m, m')) = \frac{a}{a+1}$ .

In [23], the author constructed a simple inductive limit  $W = \lim W_i'' = \lim(A(m_i, (a_i + 1)m_i), \omega_{i,j})$  such that  $K_0(W) = 0 = K_1(W)$  and  $T(W) = \{pt\}$ . In the construction, one has  $a_i + 1 = 2(a_{i-1} + 1)$  and  $m_i = a_i m_{i-1}$ . Consequently  $\lim_{i \rightarrow \infty} a_i = \infty$ . From the construction in [23], the

map  $\omega_{i,j}$  takes strictly positive elements to strictly positive elements, and  $\omega_{i,j}^*$  maps tracial state space  $T(W_j'')$  to tracial state space  $T(W_i'')$ . Furthermore,  $A_i \in \mathcal{C}_0^0$  with  $\lambda_s(A_i) = \frac{a_i}{a_{i+1}} \rightarrow 1$  as  $i \rightarrow \infty$ .

Note that  $W \otimes Q \cong W$ . Identify  $Q \otimes M_m$  and  $Q \otimes M_{a+1}$  with  $Q$ , we can identify  $A(m, (a+1)m) \otimes Q$  with  $R(\alpha, 1)$  for  $\alpha = a/(a+1)$ . Moreover,  $W = \lim(W'_n = R(\alpha_n, 1), \iota'_{W,n})$ , where  $\iota'_{W,n} : R(\alpha_n, 1) \rightarrow R(\alpha_{n+1}, 1)$  are injective. Again, we have that  $(\iota'_{W,n})^*$  takes  $T(R(\alpha_{n+1}, 1))$  to  $T(R(\alpha_n, 1))$ .

Let  $C$  be a unital AF-algebra so that  $T(C) = T$ . We write  $C = \lim_{n \rightarrow \infty} (F_n, \iota_{F,n})$ , where  $\dim(F_n) < \infty$  and  $\iota_{F,n} : F_n \rightarrow F_{n+1}$  are unital injective homomorphisms.

Let  $W$  be as before. Write

$$W_T = W \otimes C.$$

Then  $T(W_T) = T$  and  $W_T$  has continuous scale.

Suppose that

$$F_n = \bigoplus_{i=0}^{k(n)} M_{n_i}.$$

By identifying  $R(\alpha_n, 1)$  with  $R(\alpha_n, 1) \otimes M_{n_i}$  and  $R(\alpha_n, 1) \otimes Q$ , we may write that

$$W_T = \lim_{n \rightarrow \infty} (W_n, \iota_n),$$

where  $W_n$  is a direct sum of  $k(n)$  summand of  $R(\alpha_n, 1)$  :

$$W_n = \bigoplus_{i=0}^{k(n)} R(\alpha_n, 1)^{(i)},$$

where  $\alpha_1 < \alpha_2 < \dots < 1$ . Again, we have that  $\iota_n^*$  takes  $T(W_{n+1})$  to  $T(W_n)$ .

We write

$$W_n = R_{0,n} \bigoplus D_n,$$

where  $R_{0,n} = R(\alpha_n, 1)^{(0)}$  and

$$D_n = \bigoplus_{i=1}^{k(n)} R(\alpha_n, 1)^{(i)}.$$

In the case that  $W_n$  has only one summand, we understand that  $W_n = R_{0,n}$  and  $D_n = \{0\}$ . We also use

$$P_{0,n} : W_n \rightarrow R_{0,n} \text{ and } P_{1,n} : W_n \rightarrow D_n$$

for the projection map, i.e.,  $P_{0,n}(a \oplus b) = a$  and  $P_{1,n}(a \oplus b) = b$  for all  $a \in R_{0,n}$  and  $b \in D_n$ .

Consider

$$B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n)), \quad n = 1, 2,$$

Let  $r_n = \frac{1}{2^{n+1}k(n)}$ ,  $n = 1, 2, \dots$

Let us define a homomorphism  $\Psi_{n,n+1} : B_n \rightarrow B_{n+1}$  as follows.

On  $M_{(n!)^2}(A(W, \alpha_n))$  define  $\Psi_{n,n+1,A,A} : M_{(n!)^2}(A(W, \alpha_n)) \rightarrow M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$  by

$$\Psi_{n,n+1,A,A}(a) = \text{diag}(\Phi_{A_w, \alpha_n, \alpha_{n+1}}(a), \overbrace{0, 0, \dots, 0}^{((n+1)!)^2 - (n!)^2}) \text{ for all } a \in M_{(n!)^2}(A(W, \alpha_n))$$

and define  $\Psi_{n,n+1,A,W} : M_{(n!)^2}(A(W, \alpha_n)) \rightarrow R_{0,n+1} \otimes e_{r_n} Q e_{r_n}$  by

$$\Psi_{n,n+1,A,W} = q_{r_n} \circ \iota'_{W,n} \circ \widetilde{\iota^{M_{(n!)^2}}} \circ (\varphi_{A,R,\alpha_n} \otimes \text{id}_{M_{(n!)^2}}).$$

(Recall that  $\widetilde{\iota^Q} : R(\alpha, 1) \otimes Q \rightarrow R(\alpha, 1)$  is an isomorphism and  $\widetilde{\iota^{M_m}} : R(\alpha, 1) \otimes M_m \rightarrow R(\alpha, 1)$  is defined by  $\widetilde{\iota^{M_m}} = \widetilde{\iota^Q}|_{R(\alpha, 1) \otimes M_m}$ .) It is injective.

On  $W_n$  define  $\Psi_{n,n+1,W,W} : W_n \rightarrow R_{0,n+1} \otimes (1 - e_{r_n})Q(1 - e_{r_n}) \oplus D_{n+1} \subset W_{n+1}$  by, for  $a \in R_{0,n}$ ,  $b \in D_n$ ,

$$\begin{aligned} \Psi_{n,n+1,W,W}((a \oplus b)) &= \Psi_{n,n+1,W,W}^0((a \oplus b)) \oplus \Psi_{n,n+1,W,W}^1((a \oplus b)) = \\ &= q_{1-r_n}((P_{0,n+1} \circ \iota_{n,n+1}(a)) \oplus (P_{0,n+1} \circ \iota_{n,n+1}(b))) \\ &\quad \oplus (P_{1,n+1} \circ \iota_{n,n+1}(a) \oplus P_{1,n+1} \circ \iota_{n,n+1}(b)). \end{aligned} \quad (\text{e 6.36})$$

Suppose that  $a, b \geq 0$ . Then, for any  $t \in T(W_{n+1})$ ,

$$t(\Psi_{n,n+1,W,W}(a \oplus b)) \geq (1 - r_n)t(\iota_{n,n+1}(a \oplus b)). \quad (\text{e 6.37})$$

Define  $\Psi_{n,n+1,W,A} : R_{0,n} \rightarrow M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$  by

$$\Psi_{n,n+1,W,A}(a) = \text{diag}(0, \overbrace{(\varphi_{R,A,\alpha_{n+1}} \circ \iota'_{W,n})(a), \dots, (\varphi_{R,A,\alpha_{n+1}} \circ \iota'_{W,n})(a))}^{((n+1)!)^2 - (n!)^2}).$$

Now if  $(a \oplus b) \oplus c \in W_n \oplus A(W, \alpha_n)$  (with  $a \in M_{(n!)^2}(R_{0,n})$ ,  $b \in D_n$ , and  $c \in A(W, \alpha_n)$ ), define

$$\Phi_{n,n+1}((a \oplus b) \oplus c) = d \oplus c',$$

where

$$\begin{aligned} d &= \widetilde{\iota^Q}(\Psi_{n,n+1,A,W}(c) \oplus \Psi_{n,n+1,W,W}^0(a \oplus b) \oplus \Psi_{n,n+1,W,W}^1(a \oplus b)) \in W_{n+1} \\ (\Psi_{n,n+1,A,W}(c) &\in R_{0,n+1} \otimes (e_{r_n} Q e_{r_n}), \Psi_{n,n+1,W,W}^0(a \oplus b) \in R_{0,n+1} \otimes (e_{1-r_n} Q e_{1-r_n}), \text{ and} \\ \Psi_{n,n+1,W,W}^1(a \oplus b) &\in D_{n+1}) \text{ and} \end{aligned}$$

$$c' = \text{diag}(\Psi_{n,n+1,A,A}(c), \Psi_{n,n+1,W,A}(a)) \in M_{((n+1)!)^2}(A(W, \alpha_{n+1})).$$

Since all partial maps of  $\Phi_{n,n+1}$  take the strictly positive elements to the strictly positive elements in corresponding corners,  $\Phi_{n,n+1}$  itself takes strictly positive elements to strictly positive elements. This also implies that  $\Phi_{n,n+1}^*(T(B_{n+1})) \subset T(B_n)$ . Note also that  $\Phi_{n,n+1}$  maps full elements to full elements and it is injective.

Define

$$B_T = \lim_{n \rightarrow \infty} (B_n, \Phi_{n,n+1}).$$

**Remark 6.3.** In the construction above,  $C^*$  algebras  $A$  and  $Q$  are  $\mathcal{Z}$ -stable, one can also choose the homomorphism  $s : A \rightarrow Q$  and  $j : Q \rightarrow A$  to be of the form  $s' \otimes \text{id}_{\mathcal{Z}} : A \otimes \mathcal{Z} \rightarrow Q \otimes \mathcal{Z}$  and  $j' \otimes \text{id}_{\mathcal{Z}} : Q \otimes \mathcal{Z} \rightarrow A \otimes \mathcal{Z}$  respectively, when one identifies  $A \cong A \otimes \mathcal{Z}$  and  $Q \cong Q \otimes \mathcal{Z}$ . Then  $R(\alpha, 1)$ ,  $A(W, \alpha_n)$ ,  $W_n$ ,  $B_n$  are all  $\mathcal{Z}$ -stable. One can also make the map  $\Phi_{n,n+1} : B_n \otimes \mathcal{Z} \rightarrow B_{n+1} \otimes \mathcal{Z}$  to be of form of  $\Phi' \otimes \text{id}_{\mathcal{Z}}$ . In such a way, we will have that  $B_T$  is  $\mathcal{Z}$ -stable.

By section 4 of [12], one can write  $A = \lim_{n \rightarrow \infty} (A_n, \varphi_n)$ , where each  $A_n = M_{k(n)}(C(X_n))$ , where each  $X_n$  is a finite CW complex with dimension no more than 3. Let  $s : A \rightarrow Q$  be at the beginning of 6.2. Then, by the proof of 4.7 (and using 2.29) of [12], there exists a sequence of  $M_{l(n)} \subset Q$  and homomorphisms  $s_n : A_n \rightarrow M_{l(n)}$  such that, for each fixed  $m$ ,

$$\lim_{n \rightarrow \infty} s \circ \varphi_{m,\infty}(a) = \lim_{n \rightarrow \infty} s_n \circ \varphi_{m,n}(a) \text{ for all } a \in A_m. \quad (\text{e 6.38})$$

This also follows from the following. Note  $s_{*i}(G_i) = 0$ ,  $i = 0, 1$ . Since  $K_1(Q) = \{0\}$  and  $K_0(Q) = \mathbb{Q}$  which is divisible, by Theorem 3.9 of [21], for each fixed  $m$ , there exists a sequence of homomorphisms  $\psi_k : A_m \rightarrow Q$  such that  $\psi_k(A_m)$  has finite dimension and  $\lim_{k \rightarrow \infty} \psi_k(a) =$

$s \circ \varphi_{m,\infty}(a)$  for all  $a \in A_m$ . Since finite dimensional  $C^*$ -algebras are semiprojective, one also obtains (e6.38). Then for any finite set  $\mathcal{F} \subset A(W, \alpha)$  and any  $\varepsilon > 0$ , there is a  $C^*$ -algebra of the form

$$D_n' = \{(f, a) \in C([0, 1], M_{l(n)} \otimes M_{l(n)}) \oplus A_n : f(0) = s_n(a) \otimes \text{diag}(\underbrace{1, \dots, 1}_{\alpha l(n)}, 0), \\ f(1) = s_n(a) \otimes \text{diag}(\underbrace{1, \dots, 1}_{l(n)})\}$$

such that  $\mathcal{F} \subseteq_\varepsilon D_n'$ , where  $\alpha l(n)$  is an integer. Put  $D_n = D_n' \oplus W_n$ . Then that  $D_n$  is a sub-homogeneous  $C^*$ -algebras with 3-dimensional spectrum. Moreover,  $D_n \in \overline{\mathcal{D}_2}$  defined in 4.8 of [18].

Hence  $B_T$  has the decomposition rank at most three. (In fact, one can prove that  $B_T$  is an inductive limit sub-homogeneous  $C^*$ -algebras with spectrum having dimension no more than 3, but we do not need this fact.)

**Lemma 6.4.** *Suppose that  $a \in (W_n)_+$ . Then, for any integer  $k \geq 1$  and any  $t \in T(W_{n+k})$ ,*

$$t(\Psi_{n,n+k,W,W}(a)) \geq (1 - \sum_{j=0}^{k-1} r_{n+j})t(\iota_{n,n+k}(a)). \quad (\text{e6.39})$$

*Proof.* Note  $\tau \circ \Phi_{n+1,n+2}$  is in  $T(W_{n+1})$  for all  $\tau \in T(W_{n+2})$ . Thus this lemma follows from (e6.37) and induction immediately.  $\square$

**Lemma 6.5.** *Let  $n \geq 1$  be an integer. There is a strictly positive element  $e'_0 \in W_n$  with  $\|e'_0\| = 1$  such that  $\iota_{n,\infty}(e'_0)$  is a strictly positive element. Moreover, for any  $a \in (W_n)_+ \setminus \{0\}$ , there exists  $n_0 \geq n$ ,  $x_1, x_2, \dots, x_m \in W_{n_0}$  such that*

$$\sum_{i=1}^m x_i^* \iota_{n,n_0}(a) x_i = \iota_{n,n_0}(e'_0).$$

Moreover,

$$t(\iota_{n,m}(e'_0)) \geq 7/8 \text{ for all } t \in T(W_m) \text{ and for all } m \geq n_0, \\ \text{and } \tau(\iota_{n,\infty}(e'_0)) > 15/16 \text{ for all } \tau \in T(W_T).$$

*Proof.* To simplify the notation, without loss of generality, we may let  $n = 1$ . Since  $W_T$  is simple, pick a strictly positive element in  $e'_0 \in (W_1)_+$  with  $\|e'_0\| = 1$  so that  $e' = \iota_{1,\infty}(e'_0)$  is a strictly positive in  $W_T$ . By replacing  $e'_0$  by  $g(e'_0)$  for some  $g \in C_0((0, 1])_+$  we may assume that

$$\tau(e'_0) > 15/16 \text{ for all } \tau \in T(W_T).$$

There is an integer  $n'_0 \geq 1$  such that that

$$t(\iota_{1,n}(e'_0)) \geq 7/8 \text{ for all } n \geq n'_0 \text{ and } t \in T(W_n). \quad (\text{e6.40})$$

Note that this implies that

$$t(\iota_{1,n}(f_\eta(e'_0))) \geq 3/4 \text{ for all } n \geq n_0 \text{ and } t \in T(W_n) \quad (\text{e6.41})$$

whenever  $1/16 > \eta > 0$ .

Fixed  $a \in (W_1)_+ \setminus \{0\}$ . Since  $W_T$  is simple, there exists  $n_0 \geq n'_0 \geq 1$  and  $x'_1, x'_2, \dots, x'_{m'} \in W_{n_0}$  such that

$$\left\| \sum_{i=1}^{m'} (x'_i)^* \iota_{1,n_0}(a) x'_i - \iota_{1,n_0}(e'_0) \right\| < 1/128. \quad (\text{e 6.42})$$

It follows from Lemma 2.2 of [45] that there are  $y'_1, y'_2, \dots, y'_{m'} \in W_{n_0}$  such that

$$\sum_{i=1}^{m'} (y'_i)^* \iota_{1,n_0}(a) y'_i = \iota_{1,n_0}(f_\eta(e'_0)) \quad (\text{e 6.43})$$

for some  $1/16 > \eta > 0$ . By (e 6.41),  $\iota_{1,n_0}(f_\eta(e'_0))$  is full in  $W_{n_0}$ . Therefore there are  $x_1, x_2, \dots, x_m \in W_{n_0}$  such that

$$\sum_{i=1}^m x_i^* \iota_{1,n_0}(a) x_i = \iota_{1,n_0}(e'_0). \quad (\text{e 6.44})$$

□

**Proposition 6.6.**  $B_T$  is a simple  $C^*$ -algebra.

*Proof.* It suffices to show that every element in  $(B_T)_+ \setminus \{0\}$  is full in  $B_T$ . It suffices to show that every non-zero positive element in  $\cup_{n=1}^\infty \Phi_{n,\infty}(B_n)$  is full. Take  $b \in \cup_{n=1}^\infty \Phi_{n,\infty}(B_n)$  with  $b \geq 0$  and  $\|b\| = 1$ . To simplify notation, without loss of generality, we may assume that there is  $b_0 \in B_1$  such that  $\Phi_{1,\infty}(b_0) = b$ .

Write  $b_0 = b_{00} \oplus b_{0,1}$ , where  $b_{00} \in (W_1)_+$  and  $b_{0,1} \in (A(W, \alpha_1))_+$ .

First suppose that  $b_{00} \neq 0$ .

By applying 6.5, one obtains an integer  $n_0 > 1$ ,  $x_1, x_2, \dots, x_m \in W_{n_0}$  such that

$$\sum_{i=1}^m x_i^* (\iota_{1,n_0}(b_{00})) x_i = \iota_{1,n_0}(e'_0). \quad (\text{e 6.45})$$

Let  $M = \max\{\|x_i\| : 1 \leq i \leq m\}$ . The above implies that

$$t(\iota_{1,n_0}(b_{00})) \geq \frac{7}{8mM^2} \text{ for all } t \in T(W_{n_0}). \quad (\text{e 6.46})$$

Let  $P_{W,m} : B_m \rightarrow W_m$  and  $P_{A,m} : B_m \rightarrow M_{(m)!}(A(W, \alpha_m))$  be the projections ( $m \geq 1$ ). Then, by 6.4,

$$t(P_{W,n_0}(\Phi_{1,n_0}(b_{00}))) \geq t(\Psi_{1,n_0,W,W}(b_{00})) \quad (\text{e 6.47})$$

$$\geq \left(1 - \sum_{j=0}^{n_0-1} r_{1+j}\right) t(\iota_{1,n_0}(b_{00})) \quad (\text{e 6.48})$$

$$\geq \left(1 - \sum_{j=0}^{n_0-1} r_{1+j}\right) \left(\frac{7}{8mM^2}\right) \text{ for all } t \in T(W_{n_0}). \quad (\text{e 6.49})$$

It follows that  $P_{W,n_0}(\Phi_{1,n_0}(b_{00}))$  is full in  $W_{n_0}$ . Put  $b'_{00} = P_{W,n_0}(\Phi_{1,n_0}(b_{00}))$ . By applying 6.4 again, one concludes that  $P_{W,n_0+1} \circ \Phi_{n_0,n_0+1}(b'_{00})$  is full in  $W_{n_0+1}$ .

Since  $b'_{00}$  is full in  $W_{n_0}$ ,  $P_{0,n_0}(b'_{00})$  is full in  $R_{0,n_0} = R(\alpha_{n_0}, 1)$ . Since  $\varphi_{R,A,\alpha_{n+1}} \circ \iota'_{W,n}$  maps full elements of  $R_{\alpha_{n_0},1}$  to full elements in  $A(W, \alpha_{n_0+1})$ ,  $P_{A,n_0+1} \circ \Phi_{n_0,n_0+1}(b'_{00})$  is full in  $M_{(n+1)!}(A(W, \alpha_{n_0+1}))$ . It follows that  $\Phi_{n_0,n_0+1}(b'_{00})$  is full in  $B_{n_0+1}$ .

Note that what has been proved: for any  $b' \in (W_n)_+ \setminus \{0\}$ , there is  $m_0 \geq 1$  such that  $\Phi_{n,m_0}(b')$  is full in  $B_{m_0}$ . Therefore  $\Phi_{n,m}(b')$  is full in  $B_m$  for all  $m \geq m_0$ .

In particular, this shows that  $\Phi_{n,\infty}(b_{00})$  is full. Therefore  $b \geq \Phi_{n,\infty}(b_{00})$  is full.

Now consider the case that  $b_{00} = 0$ . Then  $b_{1,0} \neq 0$ . Since  $\Psi_{1,2,A,W}$  is injective,  $P_{W,1}(\Phi_{1,2}(b_{1,0})) \neq 0$ . Applying what has been proved,  $\Phi_{2,\infty}(P_{W,1}(\Phi_{1,2}(b_{1,0})))$  is full in  $B_T$ . But

$$\Phi_{1,\infty}(b_{1,0}) \geq \Phi_{2,\infty}(P_{W,1}(\Phi_{1,2}(b_{1,0}))).$$

This shows that, in all cases,  $b$  is full in  $B_T$ . Therefore  $B_T$  is simple.  $\square$

**Proposition 6.7.**  $B_T \in \mathcal{D}_0$  and  $T(B_T) = T$ . In particular,  $B_T$  has continuous scale. Moreover  $B_T$  is locally approximated by sub-homogenous  $C^*$ -algebras with spectrum having dimension no more than 3, has finite nuclear dimension,  $\mathcal{Z}$ -stable and has stable rank one.

*Proof.* Let us first show that  $T(B_T) = T$ . Recall  $\tilde{T}(A)$  is the set of all lower semi-continuous traces on  $A$  and  $T(A)$  is the set of tracial states on  $A$ . In the rest of the proof, for all  $C^*$  algebras  $A = B_n$  and  $A = W_n$ , we have that  $0 < \alpha_n \leq \inf\{d_\tau(a) : \tau \in \overline{T(A)}^w\}$ , and that all traces  $\tau \in \tilde{T}(A)$  are bounded trace.

Note the homomorphisms  $\Phi_{n,n+1} : B_n \rightarrow B_{n+1}$  and  $\iota_{n,n+1} : W_n \rightarrow W_{n+1}$  induce maps  $\Phi_{n,n+1}^* : \tilde{T}(B_{n+1}) \rightarrow \tilde{T}(B_n)$  and  $\iota_{n,n+1}^* : \tilde{T}(W_{n+1}) \rightarrow \tilde{T}(W_n)$ . From the construction above, (see also [23]), since  $\Phi_{n,n+1}$  and  $\iota_{n,n+1}$  map strictly positive elements to strictly positive elements,  $\Phi_{n,n+1}^*$  and  $\iota_{n,n+1}^*$  take tracial states to tracial states. That is,  $\Phi_{n,n+1}^* : T(B_{n+1}) \subset T(B_n)$  and  $\iota_{n,n+1}^* : T(W_{n+1}) \subset T(W_n)$ . Consequently for any  $\tau \in \tilde{T}(B_{n+1})$  (or  $\tau \in \tilde{T}(W_{n+1})$ ), we have  $\|\Phi_{n,n+1}^*(\tau)\| = \|\tau\|$  (or  $\|\iota_{n,n+1}^*(\tau)\| = \|\tau\|$ ).

Hence we have the following inverse limit systems of compact convex spaces:

$$\begin{aligned} \overline{T(B_1)}^w &\xleftarrow{\Phi_{1,2}^*} \overline{T(B_2)}^w \xleftarrow{\Phi_{2,3}^*} \overline{T(B_3)}^w \xleftarrow{\dots} \xleftarrow{\dots} \xleftarrow{\dots} \lim_{\leftarrow} \overline{T(B_n)}^w, \\ \overline{T(W_1)}^w &\xleftarrow{\iota_{1,2}^*} \overline{T(W_2)}^w \xleftarrow{\iota_{2,3}^*} \overline{T(W_3)}^w \xleftarrow{\dots} \xleftarrow{\dots} \xleftarrow{\dots} \lim_{\leftarrow} \overline{T(W_n)}^w. \end{aligned}$$

Here we write that

$$\lim_{\leftarrow} \overline{T(B_n)}^w = \{(\tau_1, \tau_2, \dots, \tau_n, \dots) \in \prod_n \overline{T(B_n)}^w : \Phi_{n,m}^*(\tau_m) = \tau_n\},$$

which is a subspace of the product space  $\prod_n \overline{T(B_n)}^w$  with product topology. On the other hand, since all the map  $\Phi_{n,m}^*$  are affine map,  $\lim_{\leftarrow} \overline{T(B_n)}^w$  has a natural affine structure defined by

$$t(\tau_1, \tau_2, \dots, \tau_n, \dots) + (1-t)(\tau'_1, \tau'_2, \dots, \tau'_n, \dots) = (t\tau_1 + (1-t)\tau'_1, t\tau_2 + (1-t)\tau'_2, \dots, t\tau_n + (1-t)\tau'_n),$$

for any  $(\tau_1, \tau_2, \dots, \tau_n, \dots), (\tau'_1, \tau'_2, \dots, \tau'_n, \dots) \in \lim_{\leftarrow} \overline{T(B_n)}^w$  and  $t \in (0, 1)$ .

Note that each element in  $\lim_{\leftarrow} \overline{T(B_n)}^w$  is given by  $(\tau_1, \tau_2, \dots, \tau_n, \dots)$  with  $\Phi_{n,m}^*(\tau_m) = \tau_n$ , for  $m > n$ . This element decides a unique element  $\tau \in \tilde{T}(B)$  defined by  $\tau|_{B_n} = \tau_n$ . However, since  $\|\tau_n\| \geq \alpha_n$  and  $\lim_n \alpha_n = 1$ ,  $\tau \in T(B_T)$ . On the other hand, each element  $\tau \in T(B_T)$  defines a sequence  $\{\tau_n = \tau|_{B_n} \in \tilde{T}(B_n)\}_n$ . Since  $\cup_n B_n$  is dense in  $B$ ,  $\|\tau\| = \lim_{n \rightarrow \infty} \|\tau_n\|$ . From  $\|\Phi_{n,n+1}^*(\tau')\| = \|\tau'\|$  for any  $\tau' \in \tilde{T}(B_{n+1})$ , we know that  $\|\tau_n\| = \|\tau_{n+1}\|$ . Consequently  $\|\tau_n\| = \|\tau\| = 1$  for all  $n$ .

Hence  $\tau_n \in T(B_n) \subset \overline{T(B_n)}^w$ . Consequently,  $T(B_T) = \lim_{\leftarrow} \overline{T(B_n)}^w$ . Similarly,  $T(W_T) = \lim_{\leftarrow} \overline{T(W_n)}^w$ . (Note that the map  $T(B_T) \rightarrow \overline{T(B_n)}^w$  from the reverse limit is the same as  $\Phi_{n,\infty}^* : T(B_T) \rightarrow \overline{T(B_n)}^w$ , the restrict map. That is,  $\tau \in T(B_T)$  corresponds to the sequence

$$(\Phi_{1,\infty}^*(\tau), \Phi_{2,\infty}^*(\tau), \dots, \Phi_{n,\infty}^*(\tau), \dots) = (\tau|_{B_1}, \tau|_{B_2}, \dots, \tau|_{B_n}, \dots).$$

In other word, the homeomorphism between  $T(B_T)$  and  $\lim_{\leftarrow} \overline{T(B_n)}^w$  also preserve the convex structure.)

Similarly, we also have the following inverse limit systems of the topological cones:

$$\tilde{T}(B_1) \xleftarrow{\Phi_{1,2}^*} \tilde{T}(B_2) \xleftarrow{\Phi_{2,3}^*} \tilde{T}(B_3) \xleftarrow{\quad} \cdots \xleftarrow{\quad} \tilde{T}(B_T) ,$$

$$\tilde{T}(W_1) \xleftarrow{i_{1,2}^*} \tilde{T}(W_2) \xleftarrow{i_{2,3}^*} \tilde{T}(W_3) \xleftarrow{\quad} \cdots \xleftarrow{\quad} \tilde{T}(W_T) .$$

(Again, the reverse limit is taking in the category of topological space in weak\* topology, but it automatically preserves cone structure)

Let  $\pi_n : B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n)) \rightarrow W_n$  be the projection and let  $\tilde{\Phi}_{n,n+1} = \Phi_{n,n+1}|_{W_n}$ , then we have the following (not commutative) diagram:

$$\begin{array}{ccccccc} B_1 & \xrightarrow{\Phi_{1,2}} & B_2 & \xrightarrow{\Phi_{2,3}} & B_3 & \xrightarrow{\Phi_{3,4}} & \cdots \\ \pi_1 \downarrow & \tilde{\Phi}_{1,2} \nearrow & \pi_2 \downarrow & \tilde{\Phi}_{2,3} \nearrow & \pi_3 \downarrow & & \\ W_1 & \xrightarrow{i_{1,2}} & W_2 & \xrightarrow{i_{2,3}} & W_3 & \xrightarrow{i_{3,4}} & . \end{array}$$

Even though the diagram is not commutative, from the constrection, it induces an approximate commuting diagram

$$\begin{array}{ccccccc} \tilde{T}(B_1) & \xleftarrow{\Phi_{1,2}^*} & \tilde{T}(B_2) & \xleftarrow{\Phi_{2,3}^*} & \tilde{T}(B_3) & \xleftarrow{\quad} \cdots \xleftarrow{\quad} & \tilde{T}(B_T) \\ \pi_1^* \uparrow & \tilde{\Phi}_{1,2}^* \nearrow & \pi_2^* \uparrow & \tilde{\Phi}_{2,3}^* \nearrow & \pi_3^* \uparrow & & \\ \tilde{T}(W_1) & \xleftarrow{i_{1,2}^*} & \tilde{T}(W_2) & \xleftarrow{i_{2,3}^*} & \tilde{T}(W_3) & \xleftarrow{\quad} \cdots \xleftarrow{\quad} & \tilde{T}(W_T) . \end{array}$$

That is

$$|(\tilde{\Phi}_{n,n+1}^*(\pi_{n+1}^*(\tau)))(g) - (i_{n,n+1}^*(\tau))(g)| \leq k(n)r_n \|g\| \|\tau\| \quad \text{for all } g \in W_n, \tau \in \tilde{T}(W_{n+1}); \text{ and}$$

$$|(\pi_n^*(\tilde{\Phi}_{n,n+1}^*(\tau)))(f) - (\Phi_{n,n+1}^*(\tau))(f)| \leq \left(\frac{1}{(n+1)^2} + k(n)r_n\right) \|f\| \|\tau\| \quad \text{for all } f \in B_n, \tau \in \tilde{T}(B_{n+1}).$$

(Note that  $k(n)r_n = \frac{1}{2^{n+1}}$ .)

Note that from the above, for  $\tau_{n+1} \in \tilde{T}(W_{n+1})$  if  $\tau_n = i_{n,n+1}^*(\tau_{n+1})$ , then

$$\|\pi_{n+1}^*(\tau_{n+1})\| \geq \left(1 - \frac{1}{2^{n+1}}\right) \|\tau_n\| \tag{e 6.50}$$

So, we have the following fact:

if  $(\tau_1, \tau_2, \dots, \tau_n, \dots) \in \Pi_n \tilde{T}(W_{n+1})$  satisfies  $\tau_n = i_{n,n+1}^*(\tau_{n+1})$ , then

$$\lim_{n \rightarrow \infty} \|\tau_n\| = \lim_{n \rightarrow \infty} \|\pi_n^*(\tau_n)\|.$$

The approximate intertwining induces an affine homeomorphisms  $\Pi : \tilde{T}(W_T) \rightarrow \tilde{T}(B_T)$  as follows.

For each  $\tau \in \tilde{T}(W_T)$ , for fixed  $n$ , we define a sequence of  $\{\sigma_{n,m}\}_{m>n} \subset \tilde{T}(B_n)$  by

$$\sigma_{n,m} = (\Phi_{n,m}^* \circ \pi_m^* \circ i_{m,\infty}^*)(\tau) \in \tilde{T}(B_n).$$



Recall that each element in  $\tilde{T}(B_n)$  is a bounded trace, whence it is a positive linear functional of  $B_n$ . From the above inequalities for approximately commuting diagram, one concludes that  $\{\sigma_{n,m}\}_{m>n}$  is a Cauchy sequence (in norm) in the dual space of  $B_n$ .

For each  $n$ , let  $\tau_n = \lim_{m \rightarrow \infty} \sigma_{n,m}$ . Evidently, from the inductive system above,  $\tilde{\Phi}_{n,n+1}^*(\tau_{n+1}) = \tau_n$ . Hence the sequence  $(\tau_1, \tau_2, \dots, \tau_n, \dots)$  determines an element  $\tau' \in \tilde{T}(B_T)$ . Let  $\Pi(\tau) = \tau'$ . From (e6.50) and the above mentioned fact, we know that  $\Pi$  preserves the norm and  $\Pi$  maps  $T(W_T)$  to  $T(B_T)$ . Moreover, it is clear that  $\Pi$  is also an affine map on  $T(W_T)$ .

We can define  $\Pi' : \tilde{T}(B_T) \rightarrow \tilde{T}(W_T)$  in exactly same way by replacing  $\Phi_{n,m}^*$  by  $i_{n,m}^*$ , replacing  $\pi_m^*$  by  $\tilde{\Phi}_{m,m+1}^*$ , and  $i_{m,\infty}^*$  by  $\Phi_{m+1,\infty}^*$ .

We now show that both  $\Pi$  and  $\Pi'$  are continuous on  $T(W_T)$  and  $T(B_T)$ , respectively. Let  $\{s_\lambda\} \subset T(W_T)$  be a net which converges to  $s \in T(W_T)$  point-wisely on  $W_T$ . Write  $s_\lambda = (s_{\lambda,1}, s_{\lambda,2}, \dots, s_{\lambda,n}, \dots)$  and  $s = (s_1, s_2, \dots, s_n, \dots)$ . Since  $s_{\lambda,n} = i_{n,n+1}^*(s_{\lambda,n+1})$  and  $s_n = i_{n,n+1}^*(s_{n+1})$ , for each  $n$ ,  $s_{\lambda,n}$  converges to  $s_n$  on  $W_n$ . Write  $\Pi(s_\lambda) = (\tau_{\lambda,1}, \tau_{\lambda,2}, \dots, \tau_{\lambda,n}, \dots)$  and  $\Pi(s) = (\tau_1, \tau_2, \dots, \tau_n, \dots)$ . Then, by the definition,

$$\tau_{\lambda,n} = \lim_{m \rightarrow \infty} \sigma_{\lambda,n,m} = \lim_{m \rightarrow \infty} (\Phi_{n,m}^* \circ \pi_m^* \circ i_{m,\infty}^*)(s_\lambda) \quad (\text{e6.51})$$

$$= \lim_{m \rightarrow \infty} (\Phi_{n,m}^* \circ \pi_m^*)(s_{\lambda,m}) \quad \text{and} \quad (\text{e6.52})$$

$$\tau_n = \lim_{m \rightarrow \infty} \sigma_{n,m} = \lim_{m \rightarrow \infty} (\Phi_{n,m}^* \circ \pi_m^* \circ i_{m,\infty}^*)(s) \quad (\text{e6.53})$$

$$= \lim_{m \rightarrow \infty} (\Phi_{n,m}^* \circ \pi_m^*)(s_m). \quad (\text{e6.54})$$

For  $b \in B_n$  and  $m > n$ ,

$$(\Phi_{n,m}^* \circ \pi_m^*)(s_{\lambda,m})(b) = s_{\lambda,m}(\pi_m \circ \Phi_{n,m}(b)) \quad \text{and} \quad (\text{e6.55})$$

$$(\Phi_{n,m}^* \circ \pi_m^*)(s_m)(b) = s_m(\pi_m \circ \Phi_{n,m}(b)). \quad (\text{e6.56})$$

Let  $\varepsilon > 0$  and let  $\mathcal{F} \subset B_n$  be a finite subset. We may assume that  $\mathcal{F}$  is in the unit ball of  $B_n$ .

There exists  $m_0 \geq 1$  such that, for all  $m \geq m_0$ ,

$$|s_{\lambda,n}(\pi_m \circ \Phi_{n,m}(b)) - \tau_{\lambda,n}(b)| < \varepsilon/3 \quad \text{and} \quad (\text{e6.57})$$

$$|s_n(\pi_m \circ \Phi_{n,m}(b)) - \tau_n(b)| < \varepsilon/3 \quad (\text{e6.58})$$

for all  $b$  in the unit ball of  $B_n$ .

Since  $s_{\lambda,n} \rightarrow s_n$  on  $B_n$  point-wisely, There exists  $\lambda_0$  such that, for all  $\lambda > \lambda_0$ ,

$$|s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b))| < \varepsilon/3 \quad (\text{e6.59})$$

for all  $b \in \mathcal{F}$ . It follows that, when  $\lambda > \lambda_0$ , for all  $b \in \mathcal{F}$ ,

$$|\tau_{\lambda,n}(b) - \tau_n(b)| \leq |\tau_{\lambda,n}(b) - s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b))| \quad (\text{e6.60})$$

$$+ |s_{\lambda,n}(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b))| \quad (\text{e6.61})$$

$$+ |s_n(\pi_{m_0} \circ \Phi_{n,m_0}(b)) - \tau_n(b)| < \varepsilon/3 + \varepsilon/3 + \varepsilon/3 = \varepsilon. \quad (\text{e6.62})$$

This verifies that  $\Pi(s_\lambda)$  converges to  $\Pi(s)$  on  $B_n$  for each  $n$ . Since  $\cup_{n=1} B_n$  is dense in  $B_T$ , it is easy to see that  $\Pi(s_\lambda)$  converges to  $\Pi(s)$  point-wisely. It follows that  $\Pi$  is weak\*-continuous on  $T(W_T)$ . A similar argument verifies that  $\Pi'$  is weak\*-continuous on  $T(B_T)$ . From the definition, one can also verify that  $\Pi$  and  $\Pi'$  are inverse each other. Consequently, they induce the homeomorphism between  $T(W_T)$  and  $T(B_T)$ . Hence  $T(B_T) = T(W_T) = T$ .

From Remark 6.3, we know that  $B_T$  is locally approximated by sub-homogenous  $C^*$ -algebras with spectrum having dimension no more than 3, has finite nuclear dimension and  $\mathcal{Z}$ -stable. It

follows from a theorem of Rørdam (see 3.5 of [17]) that  $B_T$  has strictly comparison for positive elements. Since  $T$  is compact, it follows from 9.3 of [17] that  $B_T$  has continuous scale.

It remains to show that  $B_T \in \mathcal{D}_0$ . Since  $B_T$  has continuous scale, to prove  $B_T \in \mathcal{D}_0$ . let  $a \in A_+$  be a strictly positive element with  $\|a\| = 1$ . Without loss of generality, we may assume that  $\tau(f_{1/2}(a)) \geq 15/16$  for all  $\tau \in T(B_T)$ . We choose  $a$  such that  $a = (a_a, a_w) \in B_1 = A(W, \alpha_1) \oplus W_1$  such that

$$t(a_w) > 3/4, \quad t(f_{1/2}(a_w)) > 3/4 \quad \text{for all } t \in T(W_1). \quad (\text{e 6.63})$$

Choose  $f_a = 5/16$ . Let  $b \in A_+ \setminus \{0\}$  and let  $\mathcal{F} \subset B_T$  be a finite set and  $\varepsilon > 0$ . Let  $\delta > 0$ . Without loss of generality, we may assume  $F \cup \{a, b\} \subset B_n$  for  $n$  large enough, and let  $\Lambda : B_T \rightarrow B_n$  be a completely positive contractive linear map such that

$$\|\Lambda(b) - b\| < \min\{\varepsilon/2, \delta\} \quad \text{for all } b \in \mathcal{F}. \quad (\text{e 6.64})$$

We choose  $\delta$  so small that

$$\|f_{1/2}(\Lambda(a)) - \Lambda(f_{1/2}(a))\| < 1/16 \quad \text{and} \quad \|f_{1/2}(\Lambda(a) - \Lambda(f_{1/2}(a)))\| < 1/16. \quad (\text{e 6.65})$$

Let  $P_A : B_n \rightarrow M_{(n)!^2}(A(W, \alpha_n))$  and  $P_W : B_n \rightarrow W_n$  be the canonical projections. We choose  $n \geq 1$  such that

$$\frac{1}{(n+1)^2} < \inf\{\tau(b) : \tau \in T(B_T)\}/2. \quad (\text{e 6.66})$$

We will choose the algebra  $D \in \mathcal{C}_0^0$  to be  $D = \Psi_{n,n+1,W,A}(W_n) \oplus W_{n+1}$  and the map  $\varphi : B_T \rightarrow B_T$  and  $\psi : B_T \rightarrow D$  be defined by

$$\varphi = \Phi_{n+1,\infty} \circ \Psi_{n,n+1,A,A} \circ P_A \circ \Lambda \quad \text{and}$$

$$\psi = \Psi_{n,n+1,W,A} \circ P_W \circ \Lambda \oplus \text{diag}(\Psi_{n,n+1,A,W} \circ P_A \circ \Lambda, \Psi_{n,n+1,W,W} \circ P_W \circ \Lambda).$$

Put

$$\psi' = \Psi_{n,n+1,W,A} \circ P_W \oplus \text{diag}(\Psi_{n,n+1,A,W} \circ P_A, \Psi_{n,n+1,W,W} \circ P_W) \quad (\text{e 6.67})$$

from  $B_n$  to  $D$ . Since  $\Psi_{n,n+1,W,A}$  is injective on  $W_n$ ,  $D \in \mathcal{C}_0^0$ . Since  $\Phi_{n,\infty}$  is injective, we will identify  $D$  with  $\Phi_{n,\infty}(D)$ . With this identification, we have

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon \quad \text{for all } x \in \mathcal{F}. \quad (\text{e 6.68})$$

It follows from 6.4 that

$$P_W(\Phi_{1,n}(f_{1/2}(a))) \geq \Phi_{1,n}(a_W) \quad \text{and} \quad (\text{e 6.69})$$

$$t(P_W(\Phi_{1,n}(f_{1/2}(a)))) \geq t(\Phi_{1,n}(f_{1/2}(a_W))) \geq (1 - \sum_{j=0}^{n-1} r_{1+j})t(\iota_{1,n}(f_{1/2}(a_W))) \quad (\text{e 6.70})$$

for all  $t \in T(W_n)$ . Since  $t \circ \iota_{1,n}$  is a tracial state on  $W_1$  as proved above, by (e 6.63),

$$t(P_W(\Phi_{1,n}(f_{1/2}(a)))) \geq (1/2)(3/4) = 3/8 \quad \text{for all } t \in T(W_n). \quad (\text{e 6.71})$$

Since  $\Psi_{n,n+1,W,A}$  sends strictly positive elements of  $W_n$  to those of  $\Psi_{n,n+1,W,A}(W_n)$ , any  $t' \in T(\Psi_{n,n+1,W,A}(W_n))$  gives a tracial state of  $W_n$ , therefore

$$t'(\Psi_{n,n+1,W,A}(P_W(\Phi_{1,n}(f_{1/2}(a)))) \geq 3/8 \quad \text{for all } \tau' \in T(\Psi_{n,n+1,W,A}(W_n)). \quad (\text{e 6.72})$$

For any  $t \in T(W_{n+1})$ , by applying (e 6.63) again,

$$t(\Psi_{n,n+1,W,W}(P_W(f))) \geq (1 - \sum_{j=0}^n r_{1+j})t(\iota_{1,n+1}(f_{1/2}(a_W))) \geq (1/2)(3/8) = 3/8. \quad (\text{e 6.73})$$

Combining (e 6.72) and (e 6.73), we have that

$$t(\psi'(\Phi_{1,n}(f_{1/2}(a)))) \geq t(\psi'(P_W(\Phi_{1,n}(f_{1/2}(a)))) \geq 3/8 \text{ for all } t \in T(D). \quad (\text{e 6.74})$$

It follows that, for all  $t \in T(D)$ ,

$$t(f_{1/2}(\psi(a))) \geq t(\psi'(\Phi_{1,n}(f_{1/2}(a)))) \geq t(\psi'(P_W(\Phi_{1,n}(f_{1/2}(a)))) - 1/16 \geq 5/16 = f_a. \quad (\text{e 6.75})$$

On the other hand, from the construction, for any  $c \in \Psi_{n,n+1,A,A}(M_{(n!)^2}(A(W, \alpha_n)))_+$  with  $\|c\| \leq 1$ ,

$$\tau(c) \leq \frac{1}{(n+1)^2} \text{ for all } \tau \in T(M_{((n+1)!)^2}(A(W, \alpha_{n+1}))). \quad (\text{e 6.76})$$

Therefore, for any integer  $m \geq 1$ ,

$$\tau(\varphi(a)^{1/m}) < \frac{1}{(n+1)^2} \text{ for all } \tau \in T(B_T). \quad (\text{e 6.77})$$

Consequently, by (e 6.66),

$$d_\tau(\varphi(a)) \leq \frac{1}{(n+1)^2} < \inf\{d_\tau(b) : \tau \in T(B_T)\} \quad (\text{e 6.78})$$

Since we have proved that  $B_T$  has strict comparison for positive elements, (e 6.78) implies that

$$\varphi(a) \lesssim b. \quad (\text{e 6.79})$$

It follows from 3.12, (e 6.68), (e 6.79) and e 6.75 that  $B_T \in \mathcal{D}_0$ . Since  $B_T \in \mathcal{D}_0$ , it follows from 15.5 of [17] that  $B_T$  has stable rank one. This completes the proof of this proposition.  $\square$

**Proposition 6.8.**  $K_0(B_T) = \ker \rho_{B_T} = G_0$  and  $K_1(B_T) = G_1$ .

*Proof.* Let  $I = C_0((0,1), Q \otimes Q)$  be the canonical ideal of  $A(W, \alpha_n)$ . Then the short exact sequence

$$0 \rightarrow I \rightarrow A(W, \alpha_n) \rightarrow A \rightarrow 0$$

induces six term exact sequence

$$\begin{array}{ccccc} K_0(I) & \longrightarrow & K_0(A(W, \alpha_n)) & \longrightarrow & K_0(A) \\ \uparrow & & & & \downarrow \partial \\ K_1(A) & \longleftarrow & K_1(A(W, \alpha_n)) & \longleftarrow & K_1(I). \end{array}$$

The map  $\partial : K_0(A) \rightarrow K_1(I) \cong K_0(Q \otimes Q)$  is given by  $\partial = (1 - \alpha_n)s_{*0}$  (defined by  $\partial(x) = (1 - \alpha_n)s_{*0}(x) \in \mathbb{Q}$  for all  $x \in K_0(A)$ ) as the difference of two induced homomorphisms at the end points. Note that  $K_1(I) = K_0(Q) = \mathbb{Q}$ . Then one checks  $(1 - \alpha_n)s_{*0}$  is surjective. Therefore

$K_0(A(W, \alpha_n)) = \ker \partial = \ker s_{*0} = G_0 = \ker \rho_A$  and  $K_1(A(W, \alpha_n)) \cong K_1(A) = G_1$ . Recall  $B_n = W_n \oplus M_{(n!)^2}(A(W, \alpha_n))$ . Since  $K_*(W) = \{0\}$ , we have

$$K_0(B_n) = \ker \rho_{B_n} = \ker \rho_{M_{(n!)^2}(A(W, \alpha_n))} = \ker \rho_A = G_0, \text{ and} \quad (e 6.80)$$

$$K_1(B_n) = K_1(M_{(n!)^2}(A(W, \alpha_n))) = K_1(A) = G_1. \quad (e 6.81)$$

Hence  $\Phi_{n,n+1,*} : K_*(B_n) \rightarrow K_*(B_{n+1})$  is completely decided by its partial map  $\Phi' : M_{(n!)^2}(A(W, \alpha_n)) \rightarrow M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$ . Also this partial map sending  $(f, a) \in M_{(n!)^2}(A(W, \alpha_n))$  to  $(g, \text{diag}(a, 0, \dots, 0)) \in M_{((n+1)!)^2}(A(W, \alpha_{n+1}))$  which induces identity maps on  $G_0$  and  $G_1$  at level of K-theory.  $\square$

**Lemma 6.9.** *Let  $G_0$  be a torsion free abelian group and let  $A$  be the unital AF algebra with*

$$(K_0(A), K_0(A)_+, [1]) = (\mathbb{Q} \oplus G_0, (\mathbb{Q}_+ \setminus \{0\} \oplus G_0) \cup \{(0, 0)\}, (1, 0)).$$

*Let  $\gamma : K_0(A) \rightarrow K_0(Q)$  be given by sending  $(r, x) \in \mathbb{Q} \oplus G_0$  to  $r \in \mathbb{Q} = K_0(Q)$ . Then one can write AF inductive limits  $A = \varinjlim_n (A_n, \varphi_{n,m})$  with injective  $\varphi_{n,m}$  and  $Q = \varinjlim_n (M_{l(n)}(\mathbb{C}), \psi_{n,m})$  such that there are injective homomorphisms  $s_n : A_n \rightarrow M_{l(n)}(\mathbb{C})$  to satisfy the following conditions:*

- (1)  $(s_n)_* : K_0(A_n) \rightarrow K_0(M_{l(n)}(\mathbb{C}))$  is surjective;
- (2)  $s_{n+1} \circ \varphi_{n,n+1} = \psi_{n,n+1} \circ s_n$  and the commutative diagram

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\varphi_{1,2}} & A_2 & \xrightarrow{\varphi_{2,3}} & A_3 & \xrightarrow{\varphi_{3,4}} & \dots\dots A \\ s_1 \downarrow & & s_2 \downarrow & & s_3 \downarrow & & \\ M_{l(1)} & \xrightarrow{\psi_{1,2}} & M_{l(2)} & \xrightarrow{\psi_{2,3}} & M_{l(3)} & \xrightarrow{\psi_{3,4}} & \dots\dots Q \end{array}$$

*induces  $s : A \rightarrow Q$  satisfy  $s_* = \gamma$ .*

*Proof.* By the classification theory of AF algebras due to Elliott, there is a one-sided intertwining

$$\begin{array}{ccccccc} F_1 & \xrightarrow{\varphi'_{1,2}} & F_2 & \xrightarrow{\varphi'_{2,3}} & F_3 & \xrightarrow{\varphi'_{3,4}} & \dots\dots A \\ \alpha_1 \downarrow & & \alpha_2 \downarrow & & \alpha_3 \downarrow & & \\ M_{m(1)} & \xrightarrow{\psi'_{1,2}} & M_{m(2)} & \xrightarrow{\psi'_{2,3}} & M_{m(3)} & \xrightarrow{\psi'_{3,4}} & \dots\dots Q, \end{array}$$

which induces a homomorphism  $\alpha : A \rightarrow Q$  with  $\alpha_* = \gamma$ , where all homomorphisms  $\alpha_n, \varphi'_{n,n+1}$  and  $\psi'_{n,n+1}$  are unital and injective. We need to modify the diagram to make the condition (1) holds.

We will define subsequence  $F_{k_n}$  and for each  $n$  construct a matrix algebra  $M_{l(n)}$ , unital injective homomorphisms  $s_n : F_{k_n} \rightarrow M_{l(n)}$ ,  $\xi_n : M_{l(n)} \rightarrow M_{m(k_n)}$  and  $\beta_{n-1} : M_{m(k_{n-1})} \rightarrow M_{l(n)}$  (if  $n > 1$ ) to satisfy the following conditions:

- (i):  $(s_n)_* : K_0(F_{k_n}) \rightarrow K_0(M_{l(n)})$  is surjective;
- (ii):  $\xi_n \circ s_n = \alpha_{k_n}$  and  $\beta_{n-1} \circ \alpha_{k_{n-1}} = s_n \circ \varphi'_{k_{n-1}, k_n}$ .

Let  $k_1 = 1$ . By identifying  $K_0(M_{m(k_1)})$  with  $\mathbb{Z}$ , there is a positive integer  $j|m(k_1)$  such that  $(\alpha_{k_1})_*(K_0(F_{k_1})) = j \cdot \mathbb{Z}$ . Let  $l(1) = \frac{m(k_1)}{j}$ . Choose a homomorphism  $s_1 : F_{k_1} \rightarrow M_{l(1)}$  to satisfy that  $(s_1)_* = \frac{(\alpha_{k_1})_*}{j} : K_0(F_{k_1}) \rightarrow K_0(M_{l(1)}) = \mathbb{Z}$  (which is surjective). Note that for any finite dimensional  $C^*$  algebra  $F$  and a matrix algebra  $M_k$ , a homomorphism  $\beta : F \rightarrow M_k$  is injective if and only if  $\beta_*(K_0(F)_+ \setminus \{0\}) \subset K_0(M_k)_+ \setminus \{0\}$ . Hence the injectivity of  $\alpha_{k_1}$  implies the

injectivity of  $s_1$ . Let  $\xi'_1 : M_{l(n)} \rightarrow M_{m(k_n)}$  be any unital embedding. Then  $(\xi'_1 \circ s_1)_* = \alpha_{k_1}_*$ . There is a unitary  $u \in M_{m(k_1)}$  such that  $\text{Adu} \circ \xi'_1 \circ s_1 = \alpha_{k_1}$ . Define  $\xi_1 = \text{Adu} \circ \xi'_1$  to finish the initial step  $n = 1$  for the induction.

Suppose that we have already carried out the construction until step  $n$ . There is an  $k_{n+1}$  such that

$$(\psi'_{k_n, k_{n+1}})_*(K_0(M_{m(k_n)})) \subset (\alpha_{k_{n+1}})_*(K_0(F_{k_{n+1}})) \subset K_0(M_{m(k_{n+1})}).$$

Again, there is a positive integer  $j|m(k_{n+1})$  such that

$$(\alpha_{k_{n+1}})_*(K_0(F_{k_{n+1}})) = j \cdot \mathbb{Z} \subset \mathbb{Z}(= K_0(M_{m(k_{n+1})})).$$

Let  $l(n+1) = \frac{m(k_{n+1})}{j}$ . As what we have done in the case for  $k_n = k_1$ , there are two injective unital homomorphisms  $s_{n+1} : F_{k_{n+1}} \rightarrow M_{l(n+1)}$  and  $\xi_{n+1} : M_{l(n+1)} \rightarrow M_{m(k_{n+1})}$  such that  $\xi_{n+1} \circ s_{n+1} = \alpha_{k_{n+1}}$ . Note that  $\xi_{n+1}$  has to be injective as  $M_{l(n+1)}$  is simple. Since the map  $(\psi'_{k_n, k_{n+1}})_* : K_0(M_{m(k_n)}) \rightarrow K_0(M_{m(k_{n+1})})$  factors through  $K_0(M_{l(n+1)})$  by  $(\xi_{n+1})_*$ , one can find a homomorphism  $\beta'_n : M_{m(k_n)} \rightarrow M_{l(n+1)}$  such that  $(\xi_{n+1})_* \circ (\beta'_n)_* = (\psi'_{k_n, k_{n+1}})_*$ . Since  $(\xi_{n+1})_*$  is injective, we know that  $(\beta'_n \circ \alpha_{k_n})_* = (s_{n+1} \circ \varphi'_{k_n, k_{n+1}})_*$ . Hence we can choose a unitary  $u \in M_{l(n+1)}$  such that  $\text{Adu} \circ \beta'_n \circ \alpha_{k_n} = s_{n+1} \circ \varphi'_{k_n, k_{n+1}}$ . In particular,  $\beta'_n$  is injective. Choose  $\beta_n = \text{Adu} \circ \beta'_n$ , we conclude that the inductive construction of  $F_{k_n}$ ,  $M_{l(n)}$ ,  $s_n : F_{k_n} \rightarrow M_{l(n)}$ ,  $\xi_n : M_{l(n)} \rightarrow M_{m(k_n)}$  and  $\beta_{n-1} : M_{m(k_{n-1})} \rightarrow M_{l(n)}$  to satisfy (i) and (ii) for all  $n$ . (Warning: we do not require that  $\xi_n \circ \beta_{n-1} = \psi'_{k_{n-1}, k_n}$ .)

Finally, let  $A_n = F_{k_n}$ ,  $\varphi_{n,n+1} = \varphi'_{k_n, k_{n+1}}$  and  $\psi_{n,n+1} : M_{l(n)} \rightarrow M_{l(n+1)}$  be defined by  $\psi_{n,n+1} = \beta_n \circ \xi_n$ . Therefore both  $\varphi_{n,n+1}$  and  $\psi_{n,n+1}$  are injective. Then

$$s_{n+1} \circ \varphi_{n,n+1} = \beta_n \circ \alpha_{k_n} = \beta_n \circ \xi_n \circ s_n = \psi_{n,n+1} \circ s_n.$$

Since  $m(k_n)|l(n+1)$ , we have  $\lim(M_{l(n)}, \psi_{n,m}) = \mathbb{Q}$ . □

**Lemma 6.10.** *Let  $G_0$  be torsion free and  $A$  be the AF algebra as in 6.9 with  $K_0(A) = \mathbb{Q} \oplus G_0$ . Let  $a$  be an positive integer and  $\alpha = \frac{a}{a+1}$ . Let  $A(W, \alpha)$  be defined in 6.2. Then  $A(W, \alpha)$  is an inductive limit of a sequence of  $C^*$ -algebras  $C_n \in \mathcal{C}_0$  with  $\lambda_s(C_n) = \alpha$  and with injective connecting maps.*

*Proof.* Let  $s : A \rightarrow \mathbb{Q}$  be as in 6.9. By Lemma 6.1,  $W(A, \alpha)$  is isomorphic to the  $C^*$ -subalgebra of  $C([0, 1], \mathbb{Q} \otimes M_{a+1}) \oplus A$  defined by

$$C = \{(f, x) \in C([0, 1], \mathbb{Q} \otimes M_{a+1}) \oplus A : \begin{array}{l} f(0) = s(x) \otimes \text{diag}(\overbrace{1, \dots, 1}^a, 0), \\ f(1) = s(x) \otimes \text{diag}(\overbrace{1, \dots, 1}^a, 1) \end{array}\}.$$

Let  $A = \lim_n (A_n, \varphi_{n,m})$  with injective  $\varphi_{n,m}$ ,  $\mathbb{Q} = \lim_n (M_{l(n)}(\mathbb{C}), \psi_{n,m})$ , and  $s_n : A_n \rightarrow M_{l(n)}(\mathbb{C})$  be described as in 6.9. Evidently  $C$  is an inductive limit of

$$C_n = \{(f, x) \in C([0, 1], M_{l(n)}(\mathbb{C}) \otimes M_{a+1}) \oplus A_n : \begin{array}{l} f(0) = s_n(x) \otimes \text{diag}(\overbrace{1, \dots, 1}^a, 0), \\ f(1) = s_n(x) \otimes \text{diag}(\overbrace{1, \dots, 1}^a, 1) \end{array}\},$$

with connecting homomorphism  $\Phi_{n,n+1} : C_n \rightarrow C_{n+1}$  be given by

$$\Phi_{n,n+1}(f, x) = (g, y) \quad \text{for } (f, x) \in C_n,$$

where  $g(t) = (\psi_{n,n+1} \otimes id_{a+1})(f(t))$  and  $y = \varphi_{n,n+1}(x)$ . Since both  $\varphi_{n,n+1}$  and  $\psi_{n,n+1}$  are injective, so is  $\Phi_{n,n+1}$ . The short exact sequence

$$0 \rightarrow C_0((0,1), M_{l(n)}(\mathbb{C}) \otimes M_{a+1}) \rightarrow C_n \rightarrow A_n \rightarrow 0$$

induces the six term exact sequence of K-theory. Since  $(s_n)_{*0} : K_0(A_n) \rightarrow K_0(M_{l(n)}(\mathbb{C}))$  is surjective, exactly as the beginning of proof of Proposition 6.8, we have  $K_0(C_n) = \ker((s_n)_{*0}) \subset K_0(A_n)$  and  $K_1(C_n) = 0$ . From a standard calculation (see section 3 of [18]), we know that  $K_0(C_n)_+ = \ker(s_n)_{*0} \cap K_0(A_n)_+$ . On the other hand, since  $s_n$  is injective,  $\ker(s_n)_{*0} \cap K_0(A_n)_+ = \{0\}$ . In fact, if  $x \in \ker(s_n)_{*0} \cap K_0(A_n)_+ \setminus \{0\}$ , then there exists a projection  $p \in M_r(A_n)$  such that  $[p] = x$ . However, since  $s_n$  is injective,  $s_n(p) = q$  is a non-zero projection in  $M_r(M_{l(n)})$  which is a non-zero element in  $K_0(M_{l(n)})$ , whence  $x \notin \ker((s_n)_{*0})$ . This proves that  $K_0(C_n)_+ = \{0\}$ . Thus  $C_n \in \mathcal{C}_0$ . Since  $s_n$  are unital, from the very definition (see Definition 3.5), we have  $\lambda_s(C_n) = \alpha$ . □

Summarize the above, we get the following main theorem of this section:

**Theorem 6.11.** *Let  $G_0, G_1$  be any countable abelian groups,  $T$  be any compact metrizable Choquet simplex, then there is a simple  $C^*$ -algebra  $B \in \mathcal{D}_0$  with continuous scale such that  $K_0(B) = \ker(\rho_B) = G_0$ ,  $K_1(B) = G_1$  and  $T(B) = T$ .*

*Furthermore, if, in addition,  $G_0$  is torsion free and  $G_1 = 0$ , then  $B = \lim_{n \rightarrow \infty} (C_n, \iota_n)$  with each  $C_n \in \mathcal{C}_0$ , and  $\iota_n$  map strictly positive elements to strictly positive elements. Moreover,  $B$  is locally approximated by  $C^*$ -algebras in  $\mathcal{C}_0$ .*

*Proof.* We only need to prove the additional part. But at this case, by Lemma 6.10, we know all  $B_n$  in the construction of inductive limit of  $B$  in 6.2 are inductive limits of  $C^*$ -algebras in  $\mathcal{C}_0$  with injective connecting maps. Therefore  $B$  is locally approximated by  $C^*$ -algebras in  $\mathcal{C}_0$ . Therefore  $B \in \mathcal{D}$ . Since the  $C^*$ -algebras in  $\mathcal{C}_0$  are semi-projective,  $B$  itself is an inductive limit of  $C^*$ -algebras in  $\mathcal{C}_0$ . □

**Corollary 6.12.** *Let  $G_0, G_1$  be any countable abelian groups. Let  $\tilde{T}$  be a topological cone with a base  $T$  which is a metrizable Choquet simplex and let  $\gamma : T \rightarrow (0, \infty]$  be a lower semi-continuous function and  $\tilde{\gamma} : \tilde{T} \rightarrow [0, \infty]$  be the extension of  $\gamma$  defined by  $\tilde{\gamma}(s\tau) = s\gamma(\tau)$  for any  $s \in \mathbb{R}_+$  and  $\tau \in T$ . Then there exist a non-unital simple  $C^*$ -algebra  $A$ , which is stably isomorphic to a  $C^*$ -algebra with the form  $B_T$  (in 6.7) which is in  $\mathcal{D}_0$  such that*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A, \rho_A) \cong (G_0, G_1, \tilde{T}, \tilde{\gamma}, 0)$$

*(Note that  $\rho_A = 0$  is equivalent to  $K_0(A) = \ker(\rho_A)$ .)*

*Proof.* Let  $B$  be the  $C^*$ -algebra in 6.11 with  $K_0(B) = \ker(\rho_B) = G_0$ ,  $K_1(B) = G_1$  and  $T(B) = T$ . There is a positive element (see 6.2.1 of [43], for example)  $a \in B \otimes \mathcal{K}$  such that  $d_\tau(a) = \gamma(\tau)$  for all  $\tau \in T = T(B)$ . Let  $A = \overline{a(B \otimes \mathcal{K})a}$ . Then  $A$  is stably isomorphic to  $B \in \mathcal{D}_0$  and

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A, \rho_A) \cong (G_0, G_1, \tilde{T}, \tilde{\gamma}, 0).$$

□

**Remark 6.13.** We would like to recall the following facts:

Let  $A$  be a separable  $C^*$ -algebra with  $T(A) \neq \emptyset$ . Then  $T(A)$  forms a base for the cone  $\tilde{T}(A)$ . It follows from 3.3 of [37] and 3.1 of [38] that  $\tilde{T}(A)$  forms a vector lattice. Therefore, if  $T(A)$  is compact, then  $T(A)$  is always a metrizable Choquet simplex.

**Definition 6.14.** In what follows we will use  $\mathcal{B}_T$  for the class of  $C^*$ -algebras with the form  $B_T$ . Note that if  $A \in \mathcal{B}_T$  the  $A$  is  $\mathcal{Z}$ -stable with weakly unperforated  $K_0(A)$  (see 5.5).

## 7 $C^*$ -algebras $\mathcal{Z}_0$ and class $\mathcal{D}_0$

**Definition 7.1.** Let  $\mathcal{Z}_0 = B_T$  be as constructed in the previous section with  $G_0 = \mathbb{Z}$  and  $G_1 = \{0\}$  and with unique tracial state. Note also  $\mathcal{Z}_0$  is  $\mathcal{Z}$ -stable

From Theorem 6.11 and Corollary 6.12, we have the following fact.

**Proposition 7.2.**  $\mathcal{Z}_0$  is locally approximated by  $C^*$ -algebras in  $\mathcal{C}_0$ . In fact that  $\mathcal{Z}_0 = \lim_{n \rightarrow \infty} (C_n, \iota_n)$ , where each  $C_n \in \mathcal{C}_0$ ,  $\iota_n$  maps strictly positive elements to strictly positive elements.

**Lemma 7.3.** Let  $A$  be a separable simple  $C^*$ -algebra with continuous scale. Then  $A \otimes \mathcal{Z}_0$  also has continuous scale and  $A \otimes \mathcal{Z}_0$  is  $\mathcal{Z}$ -stable.

*Proof.* Since  $\mathcal{Z}_0$  is  $\mathcal{Z}$ -stable, so is  $A \otimes \mathcal{Z}_0$ . Therefore, by [47],  $A \otimes \mathcal{Z}_0$  is purely infinite or is stably finite. Since every separable purely infinite simple  $C^*$ -algebra has continuous scale ([25]), we assume that  $A \otimes \mathcal{Z}_0$  is stably finite. In particular,  $A \otimes \mathcal{Z}$  must be stably finite and  $QT(A) \neq \emptyset$ . Since  $\mathcal{Z}$  is unital, it is easy to see that  $A \otimes \mathcal{Z}$  has continuous scale. It follows that  $T(B)$  is compact. Since  $\mathcal{Z}_0$  has a unique tracial state,  $T(B \otimes \mathcal{Z}_0)$  is also compact. The lemma follows if we also assume that  $A$  is exact by 9.3 of [17].

For general cases, let  $B = A \otimes \mathcal{Z}$ . We may write  $A \otimes \mathcal{Z}_0 = B \otimes \mathcal{Z}_0$ . We also note that  $B$  has strict comparison in the sense of 3.5 of [17].

Let  $\{e_n\}$  be an approximate identity for  $B$  such that  $e_{n+1}e_n = e_n e_{n+1} = e_n$ ,  $n = 1, 2, \dots$ . Let  $\{c_n\}$  be an approximate identity for  $\mathcal{Z}_0$  such that  $c_{n+1}c_n = c_n c_{n+1} = c_n$ ,  $n = 1, 2, \dots$ . It follows that  $a_n = e_n \otimes b_n$  is an approximate identity for  $B \otimes \mathcal{Z}_0$  such that

$$c_{n+1}c_n = (e_{n+1}e_n) \otimes (b_{n+1}b_n) = e_n \otimes b_n = c_n, \quad n = 1, 2, \dots \quad (\text{e7.1})$$

Fix any  $d \in B \otimes \mathcal{Z}_0$ . Put

$$\sigma = \inf\{d_\tau(d) : \tau \in QT(B \otimes \mathcal{Z}_0)\} > 0. \quad (\text{e7.2})$$

Since  $B$  has continuous scale, there exists an integer  $n_0 \geq 1$  such that

$$\tau(e_n - e_m) < \sigma/4 \quad \text{for all } \tau \in QT(B) \quad (\text{e7.3})$$

when  $n > m \geq n_0$ . Let  $t_Z$  be the unique tracial state of  $\mathcal{Z}_0$ . There is  $n_1 \geq 1$  such that

$$t_Z(b_n - b_m) < \sigma/4 \quad \text{for all } n > m \geq n_1. \quad (\text{e7.4})$$

We have, for  $n > m \geq n_0 + n_1$ ,

$$c_n - c_m = e_n \otimes b_n - e_m \otimes b_m = (e_n - e_m) \otimes b_n + (e_m \otimes b_n - e_m \otimes b_m) \quad (\text{e7.5})$$

$$= (e_n - e_m) \otimes b_n + (e_m \otimes (b_n - b_m)) \quad (\text{e7.6})$$

Therefore, for  $n > m \geq n_0 + n_1$ ,

$$(\tau \otimes t_Z)(c_n - c_m) < \sigma/2 \quad \text{for all } \tau \in QT(B). \quad (\text{e7.7})$$

By the strict comparison for positive element, the above inequality implies that  $c_n - c_m \lesssim d$ . It follows that  $A \otimes \mathcal{Z}_0$  has continuous scale.  $\square$

Now we are ready to state the following theorem which is a variation of 6.12:

**Theorem 7.4.** For any separable finite simple amenable  $C^*$ -algebra  $A$ , there is a  $C^*$ -algebra  $B$  which is stably isomorphic to a  $C^*$ -algebra of the form  $B_T$  in  $\mathcal{D}_0$  such that  $\text{Ell}(B) \cong \text{Ell}(A \otimes \mathcal{Z}_0)$



*Proof.* Note that, by 6.2.3 of [43], one may write

$$Cu^\sim(\mathcal{Z}_0) = \mathbb{Z} \sqcup \text{LAff}_+^\sim(\tilde{T}(\mathcal{Z}_0)) \text{ and } Cu^\sim(W) = \text{LAff}_+^\sim(\tilde{T}(W)). \quad (\text{e7.8})$$

Since both  $\mathcal{Z}_0$  and  $W$  are monotracial,  $\text{LAff}_+^\sim(\tilde{T}(\mathcal{Z}_0)) = \text{LAff}_+^\sim(\tilde{T}(W))$ . Since  $K_0(\mathcal{Z}_0) = \ker_{\rho_{\mathcal{Z}_0}}$ , one has an ordered semi-group homomorphism  $\Lambda : \mathbb{Z} \sqcup \text{LAff}_+^\sim(\tilde{T}(\mathcal{Z}_0)) \rightarrow \text{LAff}_+^\sim(\tilde{T}(W))$  which maps  $\mathbb{Z}$  to zero and identity on  $\text{LAff}_+^\sim(\tilde{T}(\mathcal{Z}_0)) = \mathbb{R}_+^\sim$ . In particular,  $\Lambda$  maps 1 to 1. It follows from 7.2 and [43] that there is a homomorphism  $\varphi_{z,w} : \mathcal{Z}_0 \rightarrow W$  which maps strictly positive elements to strictly positive elements. Let  $t_Z$  and  $t_W$  be the unique tracial states of  $\mathcal{Z}_0$  and  $W$ , respectively. Then  $t_W \circ \varphi_{z,w} = t_Z$ , since  $\mathcal{Z}_0$  has only one tracial state.

Let  $a \in P(A)_+$  be such that  $\overline{aAa}$  has continuous scale (see [28]). Put  $B = \overline{aAa} \otimes \mathcal{Z}_0$ . It is easy to verify that  $B$  is a hereditary  $C^*$ -subalgebra of  $A \otimes \mathcal{Z}_0$ . Every tracial state of  $B$  has the form  $\tau_a \otimes t_Z$ , where  $\tau \in T(\overline{aAa})$ . Fix  $\tau \in T(\overline{aAa})$ , then

$$(\tau \otimes t_z)(a \otimes z) = \tau(a)t_Z(z) = \tau(a)(t_W \circ \varphi_{z,w}(z)) \text{ for all } a \in A \text{ and } z \in \mathcal{Z}_0. \quad (\text{e7.9})$$

Let  $\psi = \text{id}_A \otimes \varphi_{z,w} : A \otimes \mathcal{Z}_0 \rightarrow A \otimes W$  and let  $s = \tau \otimes t_z \in T(B)$ . Then, by (e7.9),  $s = (\tau \otimes t_W) \circ \varphi$ . Since  $\mathcal{Z}_0$  satisfies the UCT, by Künneth formula ([44]),  $K_i(\overline{aAa} \otimes W) = 0$ ,  $i = 0, 1$ . Therefore, for any  $x \in K_0(B)$ ,  $s(x) = 0$ . This implies that  $\ker \rho_B = K_0(B)$ . Since  $A$  is separable, simple and  $B$  is a hereditary  $C^*$ -subalgebra of  $A \otimes \mathcal{Z}_0$ , by [3],  $(A \otimes \mathcal{Z}_0) \otimes \mathcal{K} \cong B \otimes \mathcal{K}$ . It follows that  $K_0(A \otimes \mathcal{Z}_0) = \ker \rho_{A \otimes \mathcal{Z}_0}$ .

Note that (see 6.13)  $T(B)$  is a metrizable Choquet simplex. By 6.12, there is a  $C^*$ -algebra  $C$  which is stably isomorphic to a  $C^*$ -algebra of the form  $B_T$  in  $\mathcal{D}_0$  such that  $\text{Ell}(C) \cong \text{Ell}(A \otimes \mathcal{Z}_0)$ .  $\square$

**Theorem 7.5.** *Let  $A$  be a separable  $C^*$ -algebra which is stably isomorphic to a  $C^*$ -algebra in  $\mathcal{D}_0$ . Then  $K_0(A) = \ker \rho_A$ .*

*Proof.* Without loss of generality, we may assume that  $A \in \mathcal{D}_0$ . By 18.3 of [17], it suffices to show every tracial state of  $A$  is a W-trace. By 18.2, it suffices to produce a sequence of completely positive contractive linear maps  $\{\varphi_n\}$  from  $A$  into  $D_n \in \mathcal{C}_0^{0'}$  such that

$$\begin{aligned} \lim_{n \rightarrow \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| &= 0 \text{ for all } a, b \in A \text{ and} \\ \tau(a) &= \lim_{n \rightarrow \infty} t_n(\varphi_n(a)) \text{ for all } a \in A, \end{aligned} \quad (\text{e7.10})$$

where  $t_n \in T(D_n)$ .

This, of course, follows directly from the definition of  $\mathcal{D}_0$ . In fact, in the proof of 13.1 of [17],  $\varphi_{1,n}$  would work (note, we assume that  $A \in \mathcal{D}_0$  instead in  $\mathcal{D}$ , therefore  $C^*$ -algebras  $D_n \in \mathcal{C}_0^{0'}$  instead in  $\mathcal{C}_0'$ ). Note also that, 13.3 of [17] shows that  $QT(Q) = T(A)$ . Thus (e7.10) follows from (e13.6) in [17].  $\square$

**Theorem 7.6.** *Let  $A$  be a separable simple  $C^*$ -algebra in  $\mathcal{D}$  with continuous scale. Then the map from  $Cu(A)$  to  $\text{LAff}_+(T(A))$  is a Cuntz semigroup isomorphism.*

*Proof.* This follows from 15.8 of [17] immediately. A direct proof could be quoted since it is identical to that of 6.2.1 of [43]. By 13.4 of [17],  $A$  has strictly comparison. Since  $A$  is stably projectionless, then the map  $\langle a \rangle \mapsto d_\tau(a)(\tau \in T(A))$  is injective. To see the surjectivity, as in the proof of 6.2.1 of [43], it suffices to show the property of almost divisibility (property (D) in the proof 6.2.1 of [43]). However, this follows easily from the fact that  $C^*$ -algebras in  $\mathcal{D}$  has a weak version of tracial approximate divisible property (14.4 of [17]).  $\square$

**Corollary 7.7.** *Let  $A$  be a separable simple  $C^*$ -algebra in  $\mathcal{D}$ . Then  $Cu^\sim(A) = K_0(A) \sqcup \text{LAff}_+^\sim(\tilde{T}(A))$ .*

*Proof.* This follows from 7.6 as exactly the same as in 6.2.3 of [43].  $\square$

**Theorem 7.8.** *Let  $A$  be a separable simple  $C^*$ -algebra in  $\mathcal{D}$  with  $\ker \rho_A = K_0(A)$ . Then  $A \in \mathcal{D}_0$ . Moreover, we have the following: There exists  $e_A \in A_+$  with  $\|e_A\| = 1$  and  $0 < \sigma_0 < 1/4$ . For any  $\varepsilon > 0$  and any finite subset  $\mathcal{F}_1 \supset \mathcal{F}$ , there are  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear maps  $\varphi : A \rightarrow A$  and  $\psi : A \rightarrow D$  for some  $C^*$ -subalgebra  $D \subset \mathcal{R}$  such that*

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon \text{ for all } x \in \mathcal{F}, \quad (\text{e 7.11})$$

$$d_\tau(\varphi(e_A)) < \eta \text{ for all } \tau \in T(A) \text{ and} \quad (\text{e 7.12})$$

$$t(f_{1/4}(\psi(e_A))) \geq 1 - \sigma_0 \text{ for all } t \in T(D). \quad (\text{e 7.13})$$

*Proof.* We may assume, without loss of generality, that  $A$  has continuous scale, by considering a hereditary  $C^*$ -subalgebra of  $A$  (see [28]). Let  $T = T(A)$ . Let  $W_T$  be the separable simple amenable  $C^*$ -algebra with  $K_0(W_T) = \{0\}$  and  $T(W_T) = T$  as in 16.4 of [17]. Therefore  $\text{LAff}_+^\sim(\tilde{T}(W_T)) = \text{LAff}_+^\sim(\tilde{T}(A))$ . Let  $\Gamma : \text{LAff}_+^\sim(\tilde{T}(W_T)) \rightarrow \text{LAff}_+^\sim(\tilde{T}(A))$  be the order semi-group isomorphism. Note that, by 7.7,  $Cu^\sim(A) = K_0(A) \sqcup \text{LAff}_+^\sim(\tilde{T}(A))$ . Since  $K_0(A) = \ker \rho_A$ , the map  $\Gamma^{-1'} : Cu^\sim(A) \rightarrow Cu^\sim(W_T)$  which maps  $K_0(A)$  to zero and  $\Gamma^{-1'}|_{\text{LAff}_+^\sim(\tilde{T}(A))} = \Gamma^{-1}$  is an order semi-group homomorphism.

Let  $1/4 > \sigma_0 > 0$ . Fix a strictly positive element  $e_A \in A_+$  with  $\|e_A\| = 1$  such that  $\tau(e_A) \geq 1 - \sigma_0/64$  for all  $\tau \in T(A)$ . Suppose also that  $\tau(f_{1/2}(e_A)) > 1 - \sigma_0/32$  for all  $\tau \in T(A)$ . Let  $\mathfrak{f} = 1 - \sigma_0/8$ . Let  $\varepsilon > 0$  and let  $\mathcal{F} \subset A$  be a finite subset. Let  $1/4 > \eta > 0$ .

Fix  $0 < \varepsilon_1 < \varepsilon/2$  and a finite subset  $\mathcal{F}_1 \supset \mathcal{F}$ . There are  $\mathcal{F}_1$ - $\varepsilon_1$ -multiplicative completely positive contractive linear maps  $\varphi : A \rightarrow A$  and  $\psi : A \rightarrow D$  for some  $C^*$ -subalgebra  $D \subset A$  such that

$$\|x - \text{diag}(\varphi(x), \psi(x))\| < \varepsilon_1/4 \text{ for all } x \in \mathcal{F}_1 \cup \{e_A\}, \quad (\text{e 7.14})$$

$$D \in \mathcal{C}'_0, \quad d_\tau(\varphi(e_A)) < \eta \text{ for all } \tau \in T(A) \text{ and} \quad (\text{e 7.15})$$

$$t(f_{1/4}(\psi(e_A))) \geq 1 - \sigma_0/8 \text{ for all } t \in T(D). \quad (\text{e 7.16})$$

Let  $\iota_D : D \rightarrow A$  be the embedding. Consider  $\Gamma^{-1'} \circ Cu(\iota_D)$ . Then, by [43], there exists a homomorphism  $\psi_1 : D \rightarrow W_T$  such that  $Cu(\psi_1) = \overline{\Gamma^{-1'} \circ Cu(\iota_D)}$ . Let  $e_d \in D$  be a strictly positive element of  $D$  with  $\|e_d\| = 1$  and let  $W_1 = \overline{\psi_1(e_d)W_T\psi_1(e_d)}$ . By [43] again, there exists a homomorphism  $\psi_{w,a} : W_T \rightarrow A$  which maps strictly positive elements to strictly positive elements and  $Cu(\psi_{w,a}) = \Gamma$ . Note that  $Cu(\iota_D) = Cu(\psi_{w,a} \circ \psi_1)$ . Note also that  $W_1 = \overline{\bigcup_{n=1}^\infty F_n \otimes W \otimes \mathbb{Q}}$ , where each  $F_n$  is finite dimensional,  $F_n \subset F_{n+1}$  and  $1_{F_n} = 1_{F_{n+1}}$ ,  $n = 1, 2, \dots$

Since  $D$  is weakly semi-projective, there exists a homomorphism  $\psi_{0,n} : D \rightarrow F_n \otimes W \otimes 1_{M_n!}$  for all large  $n$ , such that

$$\lim_{n \rightarrow \infty} \|\psi_{0,n}(g) - \psi_{w,a} \circ \psi_1(g)\| = 0 \text{ for all } g \in D. \quad (\text{e 7.17})$$

Note that  $\tau(\psi_{w,a} \circ \psi_1(f_{1/4}(\varphi(e_A)))) = \tau(f_{1/4}(\psi(e_A)))$  for all  $\tau \in T(A)$ . By passing to a subsequence, applying a weak\*-compactness argument, if necessarily, we may assume that, for all sufficiently large  $n$ ,

$$t(\psi_{w,a} \circ \psi_1(f_{1/4}(\psi(e_A)))) > 1 - \sigma_0/4 \text{ for all } t \in T(D'_{0,n}), \quad (\text{e 7.18})$$

where  $D'_{0,n} = \psi_{0,n}(F_n \otimes W \otimes M_n!)$ . Therefore

$$t(f_{1/4}(\psi_{w,a} \circ \psi_1(\psi(e_A)))) > 1 - \sigma_0/4 \text{ for all } t \in T(D'_{0,n}). \quad (\text{e 7.19})$$

Since  $Cu(\iota_D) = Cu(\psi_{w,a} \circ \psi_1)$ , by [43] again, there exists a sequence of unitaries  $u_n \in \tilde{A}$  such that

$$\lim_{n \rightarrow \infty} \|\iota_D(g) - u_n^*(\psi_{w,a} \circ \psi_1(g))u_n\| = 0 \text{ for all } g \in D. \quad (\text{e 7.20})$$

Let  $\delta > 0$  and let  $\mathcal{G} \subset D$  be a finite subset. Let  $e_n = \psi_{0,n}(e_d)$ . Choose  $1/4 > \sigma > 0$  such that

$$\|f_\sigma(e_d)gf_\sigma(e_d) - g\| < \delta/2 \text{ for all } g \in \mathcal{G}. \quad (\text{e 7.21})$$

By (e 7.17) and (e 7.20), with sufficiently small  $\delta$ , by Prop.1 of [6], there is  $n_0 \geq 1$  and unitaries  $v_n \in \tilde{A}$ ,

$$\|\iota_D(g) - v_n^*f_\sigma(e_n)\psi_{0,n}(g)f_\sigma(e_n)v_n\| < \delta \text{ for all } g \in \mathcal{G} \text{ and } v_n^*f_\sigma(e_n)v_n \in \overline{DAD}. \quad (\text{e 7.22})$$

for all  $n \geq n_0$ . Put  $\Psi : D \rightarrow A$  by  $\Psi(c) = v_{n_0}^*(\psi_{0,n_0}(c))v_{n_0}$  for all  $c \in D$ . Let  $D_0 = v_{n_0}^*D'_{0,n_0}v_{n_0}$ . by (e 7.19), we may also assume that

$$t(f_{1/4}(\Phi(\psi(e_A)))) > \mathfrak{f} \text{ for all } t \in T(D_0). \quad (\text{e 7.23})$$

Note that  $D_0 \in \mathcal{D}_0$  and  $D_0 \perp \varphi(A)A\varphi(A)$ . Moreover, with sufficiently small  $\delta$  and large  $\mathcal{G}$ ,

$$\|x - \text{diag}(\varphi(x), \Phi(\psi(x)))\| < \varepsilon \text{ for all } x \in \mathcal{F}. \quad (\text{e 7.24})$$

The lemma then follows. □

**Proposition 7.9.**  $B_T \otimes \mathcal{Z}_0 \in \mathcal{D}_0$ .

*Proof.*  $C^*$ -algebra  $B_T$  has finite nuclear dimension. It follows from 18.5 and 18.6 of [17] that  $B_T \otimes \mathcal{Z}_0$  is in  $\mathcal{D}_0$ . □

In the appendix (15.10), we will show that

**Theorem 7.10** (15.10). *Let  $A$  be a separable amenable  $C^*$ -algebra in  $\mathcal{D}$ . Then  $A \otimes \mathcal{Z} \cong A$ .*

**Definition 7.11.** By [43], there exists a homomorphism  $\varphi_{wz} : W \rightarrow \mathcal{Z}_0$  which maps the strictly positive elements to strictly positive elements, Since  $K_0(\mathcal{Z}_0) = \ker \rho_{\mathcal{Z}_0}$ , by 7.2 and by [43], there exists also a homomorphism  $\varphi_{zw} : \mathcal{Z}_0 \rightarrow W$  which maps the strictly positive elements to strictly positive elements. Note as in the proof of 7.4 we also have  $t_W = t_Z \circ \varphi_{z,w}$  and  $t_Z = t_W \circ \psi_{w,z}$ , where  $t_Z$  and  $t_W$  are the unique tracial states of  $W$  and  $\mathcal{Z}_0$  respectively.

There exists also an isomorphism  $\varphi_{w21} : M_2(W) \rightarrow W$  and an isomorphism  $\varphi_{z21} : M_2(\mathcal{Z}_0) \rightarrow \mathcal{Z}_0$  such that  $(\varphi_{z21})_{*0} = \text{id}_{K_0(\mathcal{Z}_0)}$ . We will fix these four homomorphisms.

**Definition 7.12.** Let  $\kappa_0^o : K_0(\mathcal{Z}_0) \rightarrow K_0(\mathcal{Z}_0)$  by sending  $x$  to  $-x$  for all  $x \in K_0(\mathcal{Z}_0) = \ker \rho_{\mathcal{Z}_0}$ . Denote also by  $\kappa^o$  the automorphism on  $Cu^\sim(\mathcal{Z}_0)$  such that  $\kappa^o|_{K_0(\mathcal{Z}_0)} = \kappa_0^o$  and identity on  $\text{LAff}_+(T(\mathcal{Z}_0))$  which maps function 1 to function 1. It follows from [43] that there is an endomorphism  $j^{\circ*} : \mathcal{Z}_0 \rightarrow \mathcal{Z}_0$  such that  $Cu^\sim(j^{\circ*}) = \kappa^o$  and  $j^o(a)$  is a strictly positive element of  $\mathcal{Z}_0$  for some strictly positive element  $a$ . By [43] again,  $j^{\circ*}(\mathcal{Z}_0)$  is isomorphic to  $\mathcal{Z}_0$ , say, given by  $j : j^{\circ*}(\mathcal{Z}_0) \rightarrow \mathcal{Z}_0$ . Then  $j^{\circ} = j \circ j^{\circ*}$  is an automorphism. The automorphism  $j^{\circ}$  will be also used in later sections.

**Lemma 7.13.** Define  $\Phi, \Psi : \mathcal{Z}_0 \rightarrow M_2(\mathcal{Z}_0)$  by

$$\Phi(a) = \text{diag}(a, j^{\otimes}(a)) \text{ and } \Psi(a) = (\varphi_{wz} \otimes \text{id}_{M_2})(\text{diag}(\varphi_{zw}(a), \varphi_{zw}(a))) \text{ for all } a \in \mathcal{Z}_0.$$

Then  $\Phi$  is approximately unitarily equivalent to  $\Psi$ , i.e., there exists a sequence of unitaries  $\{u_n\} \subset \tilde{M}_2(\mathcal{Z}_0)$  such that

$$\lim_{n \rightarrow \infty} \text{Ad } u_n \circ \Phi(a) = (\varphi_{wz} \otimes \text{id}_{M_2}) \circ \text{diag}(\varphi_{zw}(a), \varphi_{zw}(a)) \text{ for all } a \in \mathcal{Z}_0.$$

In particular,  $j_{*0}^{\otimes}(x) = -x$  for  $x \in K_0(\mathcal{Z}_0)$ .

Moreover  $\varphi_{z21} \circ \Phi$  is approximately unitarily equivalent to  $\varphi_{z21} \circ \Psi$ .

*Proof.* Using 6.1.1 of [43], one computes that

$$Cu^{\sim}(\Phi) = Cu^{\sim}(\Psi).$$

It follows from [43] that  $\Phi$  is approximately unitarily equivalent to  $\Psi$ . □

## 8 $\mathcal{E}(A, B)$

**Definition 8.1.** Let  $A$  be a separable amenable and let  $B$  be another  $C^*$ -algebra. We use  $B^+$  for the  $C^*$ -algebra obtained by adding a unit to  $B$  (regardless  $B$  has a unit or not). We will continue to use the embedding  $\varphi_{wz} : W \rightarrow \mathcal{Z}_0$ . Without causing confusion, we will identify  $W$  with  $\varphi_{wz}(W)$  from time to time.

An *asymptotic sequential morphism*  $\varphi = \{\varphi_n\}$  from  $A$  to  $B$  is a sequence of completely positive contractive linear maps  $\varphi_n : A \rightarrow B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  satisfies the following

- (1)  $\lim_{n \rightarrow \infty} \|\varphi_n(ab) - \varphi_n(a)\varphi_n(b)\| = 0$  for all  $a, b \in A$ ;
- (2) there is  $\alpha \in \text{Hom}_{\Lambda}(\underline{K}(A), \underline{K}(B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}))$  and there are two sequences of approximate multiplicative completely positive contractive linear maps  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{B^+} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that, for any finite subset  $\mathcal{P} \in \underline{K}(A)$ , there exists  $n_0 \geq 1$  such that

$$[\varphi_n]|_{\mathcal{P}} + [h_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [h'_n]|_{\mathcal{P}} \text{ for all } n \geq n_0.$$

Let  $\varphi = \{\varphi_n\}$  and  $\psi = \{\psi_n\}$  be two asymptotic sequential morphisms from  $A$  to  $B$ . We say  $\varphi$  and  $\psi$  are equivalent and write  $\varphi \sim \psi$  if there exist two sequences of approximately multiplicative completely positive contractive linear maps  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{B^+} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  and a sequence of unitaries  $u_n \in B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that

$$\lim_{n \rightarrow \infty} \|u_n^* \text{diag}(\varphi_n(a), h_n(a))u_n - \text{diag}(\psi_n(a), h'_n(a))\| = 0 \text{ for all } a \in A.$$

We denote by  $\langle \varphi \rangle$  for the equivalence class of asymptotic sequential morphisms represented by  $\varphi$ . Denote by  $\mathcal{E}(A, B)$  the set of all equivalence classes of asymptotic sequential morphisms from  $A$  to  $B$ .

If  $\varphi$  and  $\psi$  are two asymptotic sequential morphisms from  $A$  to  $B$ , we define  $\varphi \oplus \psi$  by  $(\varphi \oplus \psi)(a) = \text{diag}(\varphi(a), \psi(a))$  for all  $a \in A$ . Here we identify  $M_2(\mathcal{K})$  with  $\mathcal{K}$  in the usual way. We define  $\langle \varphi \rangle + \langle \psi \rangle = \langle \varphi \oplus \psi \rangle$ . This clearly defines an addition in  $\mathcal{E}(A, B)$ . Let  $\langle \psi \rangle \in \mathcal{E}(A, B)$  be represented by  $\{\psi_n\}$  whose images are in  $\mathbb{C} \cdot 1_{B^+} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Then, for any  $\langle \varphi \rangle \in \mathcal{E}(A, B)$ ,  $\langle \varphi \oplus \{\psi_n\} \rangle = \langle \varphi \rangle$ . In other words that  $\mathcal{E}(A, B)$  is a semigroup with zero represented by those  $\{\psi_n\}$  whose images are in  $\mathbb{C} \cdot 1_{B^+} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ .

**Definition 8.2.** Denote  $C = B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Let  $C_\infty = l^\infty(C)/c_0(C)$ . If  $\varphi = \{\varphi_n\}$  is an asymptotic sequential morphism, then we may view  $\varphi$  as a homomorphism from  $A$  to  $C_\infty$ . Two asymptotic sequential morphisms  $\varphi$  and  $\psi$  are *homotopy* if there is a homomorphism  $H : A \rightarrow C([0, 1], C_\infty)$  such that  $\pi_0 \circ H = \varphi$  and  $\pi_1 \circ H = \psi$ , where  $\pi_t : C([0, 1], C_\infty) \rightarrow C_\infty$  is the point-evaluation at  $t \in [0, 1]$ . Since we assume that  $A$  is amenable, there exists a completely positive contractive linear map  $L : A \rightarrow C([0, 1], l^\infty(C))$  such that  $\Pi \circ L = H$ , where  $\Pi : l^\infty(C) \rightarrow C_\infty$  is the quotient map. Denote by  $P_n : l^\infty(C) \rightarrow C$  the  $n$ -th coordinate map. Define  $\Phi'_n = P_n \circ L$ ,  $n = 1, 2, \dots$ . Define  $\varphi'_n = \pi_0 \circ \Phi'_n$  and  $\psi'_n = \pi_1 \circ \Phi'_n$ . Note that

$$\lim_{n \rightarrow \infty} \|\varphi_n(a) - \varphi'_n(a)\| = 0 \text{ for all } a \in A \text{ and} \quad (\text{e8.1})$$

$$\lim_{n \rightarrow \infty} \|\psi_n(a) - \psi'_n(a)\| = 0 \text{ for all } a \in A. \quad (\text{e8.2})$$

Therefore we may assume, without loss of generality, as far as in this section, that  $\varphi_n$  and  $\psi_n$  are homotopy for each  $n$ .

**Definition 8.3.** We now fixed a separable amenable  $C^*$ -algebra  $A$  satisfying the UCT with following property: There is a map  $T : A_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  and a sequence of approximately multiplicative completely positive contractive linear maps  $\varphi_n : A \rightarrow W$  such that, for any finite subset  $\mathcal{H} \subset A_+ \setminus \{0\}$ , there exists an integer  $n_0 \geq 1$  such that  $\varphi_n$  is  $T$ - $\mathcal{H}$ -full (see 7.8 of [17]) for all  $n \geq n_0$ .

*For the rest of this section  $A$  is as above.*

**Lemma 8.4.** *Let  $\{\varphi_n\}$  be an asymptotic sequential morphism from  $A$  to  $B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that the image of  $\varphi_n$  are all contained in  $B^\perp \otimes W \otimes \mathcal{K}$ . Then  $\langle \varphi_n \rangle = 0$ .*

*Proof.* Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset. Let  $T$  be given in 8.3. Write  $T(a) = (N(a), M(a))$  for all  $a \in A_+ \setminus \{0\}$ . We will apply Corollary 7.12 of [17]. Let  $\delta > 0$ ,  $\mathcal{G}$  be a finite subset,  $\mathcal{H} \subset A_+ \setminus \{0\}$  be a finite subset and let  $K \geq 1$  be an integer required by 7.12 of [17] for  $T$ .

Suppose that  $\varphi_n : A \rightarrow B^\perp \otimes W \otimes \mathcal{K}$  is a  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map. We may assume, without loss of generality, that the image of  $\varphi_n$  lies in  $M_{k(n)}(B^\perp \otimes W)$ .

Choose an asymptotic sequential morphism  $\{\psi_n\}$  from  $A$  to  $W$  given by 8.3. We may assume that  $\psi_n$  is  $\mathcal{G}$ - $\delta$ -multiplicative and is  $T$ - $\mathcal{H}$ -full. Let  $\psi_0 : W \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes W \otimes \mathcal{K}$  be a homomorphism. By replacing  $\{\psi_n\}$  by  $\{\psi_0 \circ \psi_n\}$ , we assume that the image of  $\psi_n$  are in  $\mathbb{C} \cdot 1_{B^\perp} \otimes W \otimes \mathcal{K}$ . Define  $\bar{\psi}_n : A \rightarrow M_{k(n)}(W)$  by

$$\bar{\psi}_n(a) = \text{diag}(\overbrace{\psi_n(a), \psi_n(a), \dots, \psi_n(a)}^{k(n)}) \text{ for all } a \in A$$

and define  $\Psi_n : A \rightarrow M_K(B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$  by

$$\Psi(a) = \text{diag}(\overbrace{\bar{\psi}_n(a), \bar{\psi}_n(a), \dots, \bar{\psi}_n(a)}^K) \text{ for all } a \in A.$$

By viewing  $\bar{\psi}_n$  as a map from  $A$  to  $M_{k(n)}((\mathbb{C} \cdot 1_{B^\perp}) \otimes W)$ , it is easy to check that  $\bar{\psi}_n$  is  $T$ - $\mathcal{H}$ -full in  $M_{k(n)}((\mathbb{C} \cdot 1_{B^\perp}) \otimes W)$ .

Choose a strictly positive element  $0 \leq c_0 \leq 1$  in  $W$ . Fixed any nonzero element  $0 \leq a \leq 1$  in  $A$ . For any  $1/2 > \eta > 0$ , there are  $x_i \in M_{k(n)}((\mathbb{C} \cdot 1_{B^\perp}) \otimes W)$  with  $\|x_i\| \leq M(a)$ ,  $1 \leq i \leq N(a)$ , such that  $f_\eta(c_0) = \sum_{i=1}^{N(a)} x_i^* \bar{\psi}_n(a) x_i$ . Hence, for any  $b \in M_{k(n)}(B^\perp \otimes W)_+$  with  $\|b\| \leq 1$ , and  $\varepsilon_1 > 0$ , there is  $1/2 > \eta > 0$  such that

$$\|b - b^{1/2}(1 \otimes f_\eta(c_0)^{1/2})b^{1/2}\| < \varepsilon_1. \quad (\text{e8.3})$$

Therefore (identifying  $\bar{\psi}_n(a)$  with  $1 \otimes \bar{\psi}_n(a)$ )

$$\left\| \sum_{i=1}^{N(a)} b^{1/2} (1 \otimes x_i)^* \bar{\psi}_n(a) (1 \otimes x_i) b^{1/2} - b \right\| < \varepsilon_1.$$

This shows that  $\bar{\psi}_n$  is  $T\text{-}\mathcal{H}$ -full in  $M_{k(n)}(B^+ \otimes W)$ .

Then, by 7.12 of [17], there exists a unitary  $v \in M_{(K+1)k(n)}(B^+ \otimes W) \subset M_{(K+1)k(n)}(B^+ \otimes \mathcal{Z}_0)$  such that

$$\|v^* \text{diag}(\varphi_n(a), \Psi_n(a))v - \text{diag}(\bar{\psi}_n(a), \Psi_n(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

This shows that  $\langle \varphi \rangle = \langle \{\psi_n\} \rangle$ . Since the images of  $\psi_n$  are assumed to be in  $\mathbb{C} \cdot 1_{B^+} \otimes W \otimes \mathcal{K} \subset \mathbb{C} \cdot 1_{B^+} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ ,  $\langle \{\psi_n\} \rangle = 0$ . Thus  $\langle \varphi \rangle = 0$ .  $\square$

**Proposition 8.5.**  $\mathcal{E}(A, B)$  is an abelian group.

*Proof.* Define an endomorphism  $\iota^*$  on  $B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  by

$$\iota^*(a \otimes b \otimes c) = a \otimes j^*(b) \otimes c \text{ for all } a \in B^+, b \in \mathcal{Z}_0 \text{ and } c \in \mathcal{K}$$

(see 7.12 for the definition of  $j^*$ ). Let  $\varphi = \{\varphi_n\}$  be an asymptotic sequential morphism from  $A$  to  $B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Let  $\psi_n : A \rightarrow B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  be defined by

$$\psi_n(a) = \iota^* \circ \varphi_n(a) \text{ for all } a \in A.$$

Define  $H \in B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K} \rightarrow M_2(B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K})$  by

$$H(a \otimes b \otimes c) = a \otimes (\varphi_{wz} \otimes \text{id}_{M_2})(\text{diag}(\varphi_{zw}(b), \varphi_{zw}(b)) \otimes c) \text{ for all } a \in B^+, b \in \mathcal{Z}_0 \text{ and } c \in \mathcal{K}.$$

It follows from 7.13 that there exists a sequence of unitaries  $\{u_n\} \subset \widetilde{B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}}$  such that

$$\text{Ad } u_n \circ H(c) = \lim_{n \rightarrow \infty} \text{diag}(\varphi_n(c), \psi_n(c)) \text{ for all } c \in B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}.$$

It follows that  $\{\varphi_n \oplus \psi_n\}$  is approximately unitarily equivalent to  $\{H \circ \varphi_n\}$ . By 8.4,  $\langle \varphi_n \oplus \psi_n \rangle = 0$ .  $\square$

**Definition 8.6.** Fixed  $A$  as in 8.3, we will consider  $\mathcal{E}(A, B)$  for separable  $C^*$ -algebra  $B$ , and denote  $\mathcal{E}(A, B)$  by  $\mathcal{E}_A(B)$ . Suppose that  $B$  and  $C$  are separable  $C^*$ -algebras and  $h : B \rightarrow C$  is a homomorphism. Define  $\mathcal{E}_A(h) : \mathcal{E}_A(B) \rightarrow \mathcal{E}_A(C)$  by  $\mathcal{E}_A(h)(\langle \varphi \rangle) = \langle \{h \circ \varphi_n\} \rangle$ , where  $\{\varphi_n\}$  is a representation of  $\langle \varphi \rangle$  and where we also use  $h$  for  $h^\sim \otimes \text{id}_{\mathcal{Z}_0 \otimes \mathcal{K}}$ . This gives a homomorphism from the abelian group  $\mathcal{E}_A(B)$  to the abelian  $\mathcal{E}_A(C)$ .

Clearly  $\mathcal{E}_A(\text{id}_B) = \text{id}_{\mathcal{E}_A(B)}$ . If  $D$  is another  $C^*$ -algebra and  $h_1 : C \rightarrow D$  is a homomorphism, then one checks that  $\mathcal{E}_A(h_1 \circ h) = \mathcal{E}_A(h_1) \circ \mathcal{E}_A(h)$ .

**Theorem 8.7.**  $\mathcal{E}(A, -) = \mathcal{E}_A(-)$  is a covariant functor from separable  $C^*$ -algebras to abelian groups which is homotopy invariant and stable, i.e.,  $\mathcal{E}_A(D) = \mathcal{E}_A(D \otimes \mathcal{K})$ .

*Proof.* From 8.5 and 8.6, it is a covariant functor from separable  $C^*$ -algebras to abelian groups. It is obviously stable. We will show it is homotopy invariant.

Fix a  $C^*$ -algebra  $B$ . Set  $C = B^+ \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Let  $\varphi$  and  $\psi$  be two homotopy asymptotic sequential morphisms from  $A$  to  $C$ . Let  $\delta > 0$  and  $\mathcal{G} \subset A$ .

Fix a large integer  $n$ . As discussed in 8.2, we may assume that there exists  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $L_n : A \rightarrow C([0, 1], C)$  which is such that  $\pi_0 \circ L_n = \varphi_n$  and  $\pi_1 \circ L_n = \psi_n$ .



Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset.

Let  $\mathcal{F}_1$  be a finite subset which contains  $\mathcal{F}$ . Let  $\mathcal{P} : 0 = t_0 < t_1 < \dots < t_k = 1$  be a partition such that

$$\|\pi_t \circ L_n(g) - \pi_{t_i} \circ L_n(g)\| < \varepsilon/4 \text{ for all } g \in \mathcal{F}_1 \quad (\text{e 8.4})$$

for all  $t \in [t_{i-1}, t_{i+1}]$ ,  $i = 1, 2, \dots, k$ . Put  $\gamma_i = \pi_{t_i} \circ L_n$ ,  $i = 0, 1, 2, \dots, k$ . Define  $\Phi_n, \Psi_n, \Phi'_n, \Psi'_n : A \rightarrow M_{2k+1}(C)$  as follows.

$$\Phi_n(a) = \text{diag}(\gamma_0(a), \iota^{\otimes} \circ \gamma_1(a), \gamma_1(a), \dots, \iota^{\otimes} \circ \gamma_k(a), \gamma_k(a)), \quad (\text{e 8.5})$$

$$\Psi'_n(a) = \text{diag}(\gamma_0(a), \iota^{\otimes} \circ \gamma_0(a), \gamma_1(a), \dots, \iota^{\otimes} \circ \gamma_{k-1}(a), \gamma_k(a)), \quad (\text{e 8.6})$$

$$\Phi'_n(a) = \text{diag}(\gamma_k(a), \iota^{\otimes} \circ \gamma_0(a), \gamma_0(a), \dots, \iota^{\otimes} \circ \gamma_{k-1}(a), \gamma_{k-1}(a)), \quad (\text{e 8.7})$$

$$\Psi_n(a) = \text{diag}(\gamma_k(a), \iota^{\otimes} \circ \gamma_1(a), \gamma_1(a), \dots, \iota^{\otimes} \circ \gamma_k(a), \gamma_k(a)) \quad (\text{e 8.8})$$

for all  $a \in A$ . We estimate that, by (e 8.4),

$$\|\Phi_n(g) - \Phi'_n(g)\| < \varepsilon/4 \text{ for all } g \in \mathcal{F}_1, \quad (\text{e 8.9})$$

$$\|\Psi_n(g) - \Psi'_n(g)\| < \varepsilon/4 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 8.10})$$

There is also a unitary  $u \in M_{2k+1}(C)$  such that

$$\|\text{Ad } u \circ \Phi'_n(g) - \Psi'_n(g)\| < \varepsilon/4 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 8.11})$$

It follows that

$$\|\text{Ad } u \circ \Phi_n(f) - \Psi_n(f)\| < 3\varepsilon/4 \text{ for all } f \in \mathcal{F}. \quad (\text{e 8.12})$$

Define  $\Theta : A \rightarrow M_{2k}(C)$  by

$$\Theta(a) = \text{diag}(\iota^{\otimes} \circ \gamma_1(a), \gamma_1(a), \dots, \iota^{\otimes} \circ \gamma_k(a), \gamma_k(a))$$

for all  $a \in A$ . Then (e 8.12) becomes

$$\|\text{Ad } u \circ \text{diag}(\varphi_n(g), \Theta(g)) - \text{diag}(\psi_n(g), \Theta(g))\| < 3\varepsilon/4 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 8.13})$$

On the other hand, by 7.13, there exists a homomorphism  $H : B^{\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \rightarrow B^{\perp} \otimes W \otimes \mathcal{K}$  and  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $\Lambda_n : A \rightarrow C$  such that

$$\|H \circ \Lambda_n(g) - \Theta(g)\| < \varepsilon/8 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 8.14})$$

Finally, we obtain that

$$\|\text{Ad } u \circ \text{diag}(\varphi_n(f), H \circ \Lambda_n(f)) - \text{diag}(\psi_n(f), H \circ \Lambda_n(f))\| < \varepsilon$$

for all  $f \in \mathcal{F}$ . Furthermore,  $\{H \circ \Lambda_n\}$  is an asymptotic sequential morphism whose images are in  $B^{\perp} \otimes W \otimes \mathcal{K}$ . This proves that  $\langle \varphi \rangle = \langle \psi \rangle$  (by 8.4).  $\square$

The proof of the following is essentially the same as that in 6.1.4 of [27].

**Proposition 8.8.** *If*

$$0 \rightarrow J \xrightarrow{j} D \xrightarrow{\pi} D/J \rightarrow 0$$

*is a short exact sequence of separable  $C^*$ -algebras, then*

$$\mathcal{E}(A, J) \xrightarrow{j_*} \mathcal{E}(A, D) \xrightarrow{\pi_*} \mathcal{E}(A, D/J)$$

*is exact in the middle.*



*Proof.* Suppose that  $\langle \varphi \rangle \in \mathcal{E}(A, J)$  which can be represented by an asymptotic sequential morphism  $\{\varphi_n\}$  which maps  $A$  to  $J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Then  $\pi \circ j \circ \varphi_n$  has image in  $\mathbb{C} \cdot 1_{(D/J)^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . It follows from the definition that  $\pi_* \circ j_* = 0$ .

Now assume that  $\langle \varphi \rangle \in \mathcal{E}(A, D)$  which is represented by  $\{\varphi_n\}$ . Without loss of generality, we may assume that  $\text{im } \varphi_n \in M_{k(n)}(D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$  for some sequence  $\{k(n)\}$ .

Suppose that  $\pi_*(\langle \varphi \rangle) = 0$ . Thus we may assume that there exist two asymptotic sequential morphisms  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{(D/J)^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  and a sequence of unitaries  $u_n \in ((D/J)^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})^\sim$  such that

$$\lim_{n \rightarrow \infty} \|u_n^* \text{diag}(\pi \circ \varphi_n(a), h_n(a))u_n - h'_n(a)\| = 0 \text{ for all } a \in A. \quad (\text{e 8.15})$$

Denote  $\Pi_{D/J} : ((D/J)^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})^\sim \rightarrow \mathbb{C}$  the quotient map. By multiplying a scalar multiple of identity, without loss of generality, we may assume that  $\Pi_{D/J}(u_n) = 1$ . Without loss of generality, we may assume that  $\text{im}(\varphi_n \oplus h_n), \text{im } h'_n \subset M_{K(n)}(\mathbb{C} \cdot 1_{(D/J)^\perp} \otimes \mathcal{Z}_0)$ . We may further assume that  $K(n) = 2k(n)$ . Since  $\Pi_{D/J}(u_n) = 1$ , we may view  $\text{diag}(u_n, u_n^*) \in ((D/J)^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})^\sim$ . Replacing  $u_n$  by  $\text{diag}(u_n, u_n^*)$ , we may assume that  $u_n \in U_0(((D/J)^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})^\sim)$ . Therefore, we may assume that there exists a unitary  $z_n \in U((D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})^\sim)$  such that  $\pi(z_n) = u_n$ .

By identifying  $\mathbb{C} \cdot 1_{(D/J)^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  with  $\mathbb{C} \cdot 1_{D^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  and  $\mathbb{C} \cdot 1_{J^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ , we may view  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{D^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  as well as  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{J^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ , whichever it is convenient.

Let  $\Pi : l^\infty(D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \rightarrow l^\infty(D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$  be the quotient map. Let

$$U = \{z_n\}, \quad Z = \Pi(U), \quad \Phi = \{\varphi_n\}, \quad H = \{h_n\}, \quad H' = \{h'_n\},$$

where we view  $\Phi, H, H' : A \rightarrow l^\infty(D^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$ . Then, by (8.4),

$$Z^*(\Pi(\Phi(a) \circ H(a)))Z - \Pi \circ H'(a) \in l^\infty(J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$$

for all  $a \in A$ . Since  $\Pi \circ H'(a) \in \mathbb{C} \cdot 1_{J^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ , it follows that

$$Z^*(\Pi(\Phi(a) \circ H(a)))Z \in l^\infty(J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})/c_0(J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})$$

for all  $a \in A$ . Since  $A$  is amenable, by [5], there exists a completely positive contractive linear map  $L = \{l_n\} : A \rightarrow J^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that  $\Pi \circ L = \text{Ad } U \circ (\Phi \oplus H)$ . Then

$$\lim_{n \rightarrow \infty} \|l_n(a) - z_n^*(\text{diag}(\varphi_n(a), h_n(a)))z_n\| = 0 \text{ for all } a \in A.$$

It follows that  $\langle \{l_n\} \rangle$  is an asymptotic sequential morphism in  $\mathcal{E}(A, J)$  and

$$\langle \{l_n\} \rangle = \langle \varphi_n \oplus \{h_n\} \rangle.$$

This implies that  $\langle \varphi \rangle$  is in the  $j_*(\mathcal{E}(A, J))$ . □

**Proposition 8.9.**  $\mathcal{E}_A(-)$  is split exact.

*Proof.* This is standard from 8.7 and 8.8 (see [20]). Let

$$0 \rightarrow J \xrightarrow{j} D \xrightarrow{\pi} D/J \rightarrow 0$$

be a short exact sequence of separable  $C^*$ -algebras.

Let us first assume that  $D/J$  is contractible. Then by 8.7,  $\mathcal{E}_A(D/J) = \{0\}$ . It follows from 8.8 that  $j_*$  gives surjective map from  $\mathcal{E}_A(J)$  onto  $\mathcal{E}_A(D)$ .

For  $C^*$ -algebra  $C$ , denote by  $S(C) = C_0([0, 1], C)$ . Then, by 8.7,

$$\mathcal{E}_A(D/J) = 0 = \mathcal{E}_A(S(D/J))$$

Put

$$S(D, D/J) = \{(a, b) \in D \oplus C_0([0, 1], D/J) : \pi(a) = b(0)\} \quad \text{and} \quad (\text{e 8.16})$$

$$Z(J, D) = \{x \in C([0, 1], D) : x(0) \in J\}. \quad (\text{e 8.17})$$

We have the following exact sequence:

$$0 = \mathcal{E}_A(S(D/J)) \longrightarrow \mathcal{E}_A(S(D, D/J)) \longrightarrow \mathcal{E}_A(D). \quad (\text{e 8.18})$$

Define  $\pi' : Z(J, D) \rightarrow C_0([0, 1], D/J)$  by  $\pi'(f)(t) = \pi(f)(1 - t)$  for  $t \in [0, 1]$ . Note  $\pi'(f)(1) = \pi(f)(0) = 0$  for all  $f \in Z(J, D)$ . Define  $\chi : Z(J, D) \rightarrow S(D, D/J)$  by

$$\chi(f) = (f(1), \pi'(f)) \quad \text{for all } f \in Z(J, D).$$

One obtains the short exact sequence:

$$0 \rightarrow C_0([0, 1], J) \rightarrow Z(J, D) \rightarrow S(D, D/J) \rightarrow 0.$$

This gives the following exact sequence:

$$0 = \mathcal{E}_A(C_0([0, 1], J)) \longrightarrow \mathcal{E}_A(Z(J, D)) \longrightarrow \mathcal{E}_A(S(D, D/J)). \quad (\text{e 8.19})$$

From (e 8.18) and (e 8.19), it follows that composition map  $\mathcal{E}_A(Z(J, D)) \rightarrow \mathcal{E}_A(S(D, D/J)) \rightarrow \mathcal{E}_A(D)$  is injective.

However,  $Z(J, D)$  is homotopically equivalent to  $J$ . Moreover one sees that the composition  $J \rightarrow Z(J, D) \rightarrow S(D, D/J) \rightarrow D$  coincides with  $j$ . It follows that  $j_*$  is injective.

Thus we show that, when  $D/J$  is contractible,  $j_*$  is an isomorphism from  $\mathcal{E}_A(J)$  onto  $\mathcal{E}_A(D)$ .

In general, let  $\iota : J \rightarrow S(D, D/J)$  be defined by  $\iota(b) = (b, 0)$  for  $b \in J$ . Then  $S(D, D/J)/\iota(J) \cong C_0([0, 1], D/J)$  which is contractible. So, from what has been proved,  $\iota_*$  is an isomorphism.

To see that  $\mathcal{E}_A(-)$  is split exact, consider the short exact sequence of separable  $C^*$ -algebras:

$$0 \rightarrow J \xrightarrow{j} D \xrightleftharpoons[\pi]{s} D/J \rightarrow 0.$$

By 8.8,

$$\mathcal{E}_A(J) \xrightarrow{j} \mathcal{E}_A(D) \xrightarrow{\pi} \mathcal{E}_A(D/J)$$

is exact in the middle. Since  $\pi \circ s = \text{id}_{D/J}$ , we check that  $\pi_* \circ s_* = (\text{id}_{D/J})_*$ .

It remains to show that  $j_*$  is injective. Using the exact sequence

$$\mathcal{E}_A(S(D/J)) \rightarrow \mathcal{E}_A(S(D, D/J)) \rightarrow \mathcal{E}_A(D),$$

and identifying  $\mathcal{E}_A(J)$  with  $\mathcal{E}_A(S(D, D/J))$ , we see that  $\ker j_* \subset \text{im } (\iota_1)_*$  where  $\iota_1 : S(D/J) \rightarrow S(D, D/J)$  is the embedding.

Let

$$I = \{(s(b(0)), b) \in S(D, D/J) : b \in C_0([0, 1], D/J)\},$$

where  $s$  is the split map given above. Since  $\pi \circ s = \text{id}_{D/J}$ ,  $I \cong C_0([0, 1], D/J)$  which is contractible. On the other hand,  $\text{im } \iota_1 \subset I$ . Therefore  $(\iota_1)_* = 0$ . Thus  $\ker j_* = 0$ . In other words,  $j_*$  is injective. □

## 9 An existence theorem

**Definition 9.1.** Fix  $A$  as in 8.3. We assume that  $A$  satisfies the UCT. There is a homomorphism  $\beta_A^B$  from  $\mathcal{E}_A(B)$  to  $KL(A, B)$  defined as follows.

We will identify  $KL(A, C)$  with  $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(C))$  for any separable  $C^*$ -algebra  $C$  (see [8]). Let  $\langle \varphi \rangle \in \mathcal{E}_A(B) = \mathcal{E}(A, B)$  represented an asymptotic morphism  $\{\varphi_n\}$ . Therefore there is an  $\alpha \in KL(A, B^\perp \otimes \mathcal{Z}_0)$  and there are two sequences of approximate multiplicative completely positive contractive linear maps  $h_n, h'_n : A \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that, for any finite subset  $\mathcal{P} \subset \underline{K}(A)$ ,

$$[\varphi_n \oplus h_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}} + [h'_n]|_{\mathcal{P}}.$$

for any sufficiently large  $n$ .

Consider the split short exact sequence

$$0 \rightarrow B \otimes \mathcal{Z}_0 \otimes \mathcal{K} \xrightarrow{i} B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K} \xrightarrow[\pi]{s} \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K} \rightarrow 0.$$

It gives the following split short exact sequence:

$$0 \rightarrow KL(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \xrightarrow{[i]} KL(A, B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \xrightarrow[\pi]{[s]} KL(A, \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \rightarrow 0. \quad (\text{e9.1})$$

Define  $\lambda : \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K})) \rightarrow \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{Z}_0 \otimes \mathcal{K}))$  by

$$\lambda(x) = x - [s] \circ [\pi](x) \text{ for all } x \in KL(A, B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}).$$

Note that

$$s \circ \pi \circ g_n = g_n$$

for any completely positive contractive linear map  $g_n : A \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$ . Therefore, for any fixed finite subset  $\mathcal{P} \subset \underline{K}(A)$ ,

$$\lambda \circ ([\varphi_n]|_{\mathcal{P}} + [h_n]|_{\mathcal{P}} - [h'_n]|_{\mathcal{P}}) = [\varphi_n]|_{\mathcal{P}} - [s \circ \pi \circ \varphi_n]|_{\mathcal{P}} = \lambda \circ \alpha|_{\mathcal{P}} \quad (\text{e9.2})$$

for all  $n \geq n_0(\mathcal{P})$  for some integer  $n_0(\mathcal{P})$ . If  $\{\psi_n\}$  is another representation of  $\langle \varphi \rangle$ , then, there exist two sequences of approximately multiplicative completely positive contractive linear maps  $g_n, g'_n : A \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  and a sequence of unitaries  $u_n \in B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that

$$\lim_{n \rightarrow \infty} \|u_n^* \text{diag}(\varphi_n(a), g_n(a)) u_n - \text{diag}(\psi_n(a), g'_n(a))\| = 0 \text{ for all } a \in A.$$

Thus there is an integer  $n_1(\mathcal{P}) \geq 1$  such that

$$[\varphi_n]|_{\mathcal{P}} + [g_n]|_{\mathcal{P}} = [\psi_n]|_{\mathcal{P}} + [g'_n]|_{\mathcal{P}} \text{ and} \quad (\text{e9.3})$$

$$[s \circ \pi \circ \varphi_n]|_{\mathcal{P}} + [g_n]|_{\mathcal{P}} = [s \circ \pi \circ \psi_n] + [g'_n]|_{\mathcal{P}} \text{ for all } n \geq n_1(\mathcal{P}) \quad (\text{e9.4})$$

Therefore

$$[\psi_n]|_{\mathcal{P}} - [s \circ \pi \circ \psi_n]|_{\mathcal{P}} = ([\varphi_n]|_{\mathcal{P}} + [g_n]|_{\mathcal{P}} - [g'_n]|_{\mathcal{P}}) \quad (\text{e9.5})$$

$$-([s \circ \pi \circ \varphi_n]|_{\mathcal{P}} + [s \circ \pi \circ g_n]|_{\mathcal{P}} - [s \circ \pi \circ g'_n]|_{\mathcal{P}}) \quad (\text{e9.6})$$

$$= [\varphi_n]|_{\mathcal{P}} - [s \circ \pi \circ \varphi_n]|_{\mathcal{P}} = \lambda \circ \alpha|_{\mathcal{P}} \quad (\text{e9.7})$$

for all  $n \geq \max\{n_0(\mathcal{P}), n_1(\mathcal{P})\}$ .

Thus  $\beta_A(\langle \varphi_n \rangle) = \lambda \circ \alpha$  is well defined. Consequently  $\beta_A^B$  is a well defined morphism which maps  $C^*$ -algebra  $B$  to abelian group  $\text{Hom}_\Lambda(\underline{K}(A), \underline{K}(B \otimes \mathcal{Z}_0 \otimes \mathcal{K}))$ . If  $B$  and  $C$  are two  $C^*$ -algebras and  $h : B \rightarrow C$  is a homomorphism we have the following commutative diagram:

$$\begin{array}{ccc} \mathcal{E}_A(B) & \xrightarrow{\mathcal{E}_A(h)} & \mathcal{E}_A(C) \\ \beta_A^B \downarrow & & \downarrow \beta_A^C \\ \text{Hom}_\Lambda(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K}) & \xrightarrow{[\text{h}]} & \text{Hom}_\Lambda(A, C \otimes \mathcal{Z}_0 \otimes \mathcal{K}) \end{array}$$

It follows that

$$\beta : \mathcal{E}_A(-) \rightarrow \text{Hom}_\Lambda(A, - \otimes \mathcal{Z}_0 \otimes \mathcal{K})$$

is a natural transformation.

**Theorem 9.2.** *The transformation  $\beta_A$  maps  $\mathcal{E}_A(B)$  onto  $\text{Hom}_\Lambda(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K})$  for each separable  $C^*$ -algebra  $B$ , if  $A$  satisfies the UCT.*

*Proof.* By a theorem of Higson (Theorem 3.7 of [20]), since  $\mathcal{E}_A(-)$  is a covariant functor from separable  $C^*$ -algebras to abelian groups which is homotopy invariant, stable and split exact (Section 8), there is a unique transformation

$$\alpha : KK(A, -) \rightarrow \mathcal{E}_A(-)$$

such that

$$\alpha_A([\text{id}_A]_{KK}) = \langle \text{id}_A \rangle.$$

Let  $\gamma : KK(A, -) \rightarrow KL(A, -)$  be the natural transformation induced by  $\Gamma : KK(A, B) \rightarrow KL(A, B)$ . We have

$$\beta_A \circ \alpha_A([\text{id}_A]) = [\text{id}_A]_{KL},$$

where  $\beta$  was defined in 9.1. Since  $\gamma([\text{id}_A]) = [\text{id}_A]$ , (the first  $[\text{id}_A]$  is in  $KK(A, A)$  and the second is in  $KL(A, A)$ ), by the uniqueness of Higson's theorem (3.7 of [20]),

$$\beta \circ \alpha = \gamma.$$

Since  $\gamma(KK(A, B)) = \text{Hom}_\Lambda(A, B \otimes \mathcal{Z}_0 \otimes \mathcal{K})$ , if  $A$  satisfies the UCT,  $\beta_A : \mathcal{E}_A(B) \rightarrow KL(A, B)$  must be surjective. □

**Lemma 9.3.** *Let  $B$  a non-unital and separable simple  $C^*$ -algebra with continuous scale. Let  $\varphi_0, \varphi_1, \varphi_2 : W \rightarrow M(B)/B$  be three non-zero homomorphisms. Then, for any  $\varepsilon > 0$ , and any finite subset  $\mathcal{F} \subset W$ , there exists a unitary  $U \in M_2(M(B))$  such that*

$$\|\pi(U)^* \text{diag}(\varphi_1(a), \varphi_0(a)) \pi(U) - \text{diag}(\varphi_2(a), \varphi_0(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

*Proof.* It follows from [25] that  $M(B)/B$  is simple and purely infinite.

Fix a strictly positive element  $a_W \in W$  with  $\|a_W\| = 1$ . Let  $b_0 = \varphi_0(a_W)$  and let  $B_0 = \overline{b_0(M(B)/B)b_0}$ .

Since  $W$  and  $B_0$  are both simple, there is a map  $T : W \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  such that  $\varphi_0 : W \rightarrow B_0$  is  $T$ - $W_+$   $\setminus \{0\}$ -full. Let  $W_0 = \varphi_0(W)$ . So  $b_0 \in W_0$ .

Let  $\mathcal{H} \subset W_+ \setminus \{0\}$  be a finite subset and  $K \geq 1$  be an integer as required by Cor. 7.11 of [17] for the above given  $T$ ,  $\varepsilon/2$  (in place of  $\varepsilon$ ) and  $\mathcal{F}$ .

Note that  $W \otimes Q \cong W$ . Moreover, the map from  $W$  to  $W \otimes 1$  which maps  $a$  to  $a \otimes 1$  then to  $W$  is approximately inner. To simplify notation, without loss of generality, we may assume

that  $\varphi_0 : W \rightarrow W_0 \otimes Q$  has the form  $\varphi_0(a) \otimes 1$ . Let  $e_1, e_2, \dots, e_K \in Q$  be mutually orthogonal and mutually equivalent projections such that  $\sum_{i=1}^K e_i = 1_Q$ . Define  $\varphi_{0,i} : W \rightarrow W_0 \otimes e_i$  by

$$\varphi_{0,i}(a) = \varphi_0(a) \otimes e_i \text{ for all } a \in W.$$

Put  $B_{0,1} = \overline{(b_0 \otimes e_1)M(B)/B(b_0 \otimes e_1)}$ .

Let  $b_1 = \varphi_1(a_W)$ ,  $b_2 = \varphi_2(a_W) \in M(B)/B$ . Since  $W$  is projectionless,  $\text{sp}(a_W) = [0, 1]$ . Thus, since  $W$  is simple,  $b_1$  can not be invertible in  $M(B)/B$ . By Pedersen's double orthogonal complement theorem (Theorem 15 of [39]), there is a projection  $E_1 \in M(B)/B$  such that  $1_{M(B)/B} - E_1 \neq 0$  and  $E_1 b_1 = b_1 E_1 = b_1$ . Similarly, one obtains a projection  $E_2 \in M(B)/B$  such that  $1_{M(B)/B} - E_2 \neq 0$  and  $E_2 b_2 = b_2 E_2 = b_2$ . Using the fact that  $M(B)/B$  is purely infinite simple again, one obtains a unitary  $w_1 \in M(B)/B$  such that

$$w_1^* E_2 w_1 \leq E_1.$$

Thus, without loss of generality, one may assume that  $E_2 \leq E_1$ .

Since  $M(B)/B$  is purely infinite and simple,  $E \lesssim p'_0$  for some projection  $p'_0 \in B_0$ . Thus we obtain a unital hereditary  $C^*$ -subalgebra  $B_{00} \subset M(B)/B$  such that, we may view that  $\varphi_1, \varphi_2 : W \rightarrow B_{00}$  and  $\varphi_{0,1} : W \rightarrow B_{00}$  is a  $T$ - $W_+$   $\setminus \{0\}$ -full. Moreover, we view

$$\varphi_0(a) = \text{diag}(\overbrace{\varphi_{0,1}(a), \varphi_{0,1}(a), \dots, \varphi_{0,1}(a)}^K) \text{ for all } a \in W.$$

Furthermore,  $M_{K+1}(B_{00})$  is a unital  $C^*$ -subalgebra of  $M_2(M(B)/B)$  such that  $1_{M_{K+1}(B_{00})}$  is not the unit of  $M_2(M(B)/B)$ . By applying Cor 7.11 of [17], there is a unitary  $u \in M_{K+1}(B_{00}) \subset M_2(M(B)/B)$  such that

$$\|u^*(\text{diag}(\varphi_1(a), \varphi_0(a)))u - \text{diag}(\varphi_2(a), \varphi_0(a))\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

Since  $1_{M_2} - 1_{M_{K+1}(B_{00})} \neq 0$  and  $M_2(M(B)/B)$  is purely infinite and simple, there exists a unitary  $v \in (1_{M_2} - 1_{M_{K+1}(B_{00})})(M(B)/B)(1_{M_2} - 1_{M_{K+1}(B_{00})})$  such that  $u \oplus v \in U_0(M_2(M(B)/B))$ . Thus we may assume that  $u$  is a unitary in  $U_0(M_2(M(B)/B))$ . Thus there is a unitary  $U \in M_2(M(B))$  such that  $\pi(U) = u$ . □

**9.4 (Construction of  $\varphi_W$ ).** Let  $B$  be a non-unital separable simple  $C^*$ -algebra with stable rank one, with  $T(B) \neq \emptyset$  and with continuous scale.

Let  $\{e_n\} \subset B \otimes \mathcal{Z}_0$  be an approximate identity with

$$e_{n+1}e_n = e_n e_{n+1} = e_n \text{ for all } n \in \mathbb{N}.$$

We may assume that  $e_{n+1} - e_n \neq 0$  for all  $n \geq 1$ . Choose  $k(n) \geq 1$  such that

$$\inf\{d_\tau(e_{4n} - e_{4n-1}) : \tau \in T(B \otimes \mathcal{Z}_0)\} > \frac{1}{k(n)}, \quad n = 1, 2, \dots$$

Note that  $\sum_{n=1}^\infty \frac{1}{k(n)} < 1$ .

Put  $B_n := \overline{(e_{4n} - e_{4n-1})(B \otimes \mathcal{Z}_0)(e_{4n} - e_{4n-1})}$ . Fix a strictly positive element  $a_w \in W$  with  $\|a_w\| = 1$ .

It follows from [43] that there is a homomorphism  $\varphi_{0,n} : W \rightarrow B_n$  such that

$$d_\tau(\varphi_{0,n}(a_w)) = \frac{1}{k(n)} \text{ for all } \tau \in T(B).$$

Let  $\varphi_W : W \rightarrow M(B \otimes \mathcal{Z}_0)$  be defined by

$$\varphi_{\text{even}} = \sum_{n=1}^{\infty} \varphi_{0,2n}, \quad \varphi_{\text{odd}} = \sum_{n=1}^{\infty} \varphi_{0,2n+1} \quad \text{and} \quad (\text{e 9.8})$$

$$\varphi_W = \sum_{n=1}^{\infty} \varphi_{0,n} = \text{diag}(\varphi_{\text{even}}, \varphi_{\text{odd}}). \quad (\text{e 9.9})$$

**Proposition 9.5.** *Let  $B$  be a non-unital separable simple  $C^*$ -algebra with stable rank one, with  $T(B) \neq \emptyset$  and with continuous scale. Fix an integer  $k \geq 1$ . Let  $j_{w,z} : W \rightarrow M_k(\mathcal{Z}_0)$  be an embedding which maps strictly positive elements to strictly positive elements and  $d : \mathcal{Z}_0 \rightarrow \mathbb{C} \cdot 1_{M_k(\tilde{B})} \otimes \mathcal{Z}_0 \subset M_k(\tilde{B} \otimes \mathcal{Z}_0) \subset M(M_k(B \otimes \mathcal{Z}_0))$  be the embedding defined by  $d(z) = 1 \otimes z$  for all  $z \in \mathcal{Z}_0$ .*

*Let  $\varepsilon > 0$  and  $\mathcal{F} \subset W$  be a finite subset. Then there is an integer  $K \geq 1$  and a unitary  $u \in M_{K+1}(M(M_k(B \otimes \mathcal{Z}_0)))$  such that*

$$\|u^*(d_K \circ j_{w,z}(a))u - (d_K \circ j_{w,z}(a) \oplus \varphi_{\text{odd}}(a))\| < \varepsilon \quad \text{for all } a \in \mathcal{F},$$

where

$$d_K(z) = \text{diag}(\overbrace{d(z), d(z), \dots, d(z)}^K) \quad \text{for all } z \in \mathcal{Z}_0.$$

*Proof.* The proof has the same spirit as that of 9.3. Keep in mind that  $B$  has continuous scale. Therefore  $M(M_k(B \otimes \mathcal{Z}_0))$  has only one (closed) ideal  $M_k(B \otimes \mathcal{Z}_0)$ . Since  $W$  is simple and  $d \circ j_{w,z}$  maps a strictly positive element to that of  $\mathbb{C} \cdot 1_{M_k(\tilde{B})} \otimes \mathcal{Z}_0$  which is not in  $M_k(B \otimes \mathcal{Z}_0)$ ,  $d \circ j_{w,z}(W)$  is full in  $M(M_k(B \otimes \mathcal{Z}_0))$ . There is a map  $T : W_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  such that  $d \circ j_{w,z}$  is  $T$ - $W_+ \setminus \{0\}$ -full in  $M(M_k(B \otimes \mathcal{Z}_0))$ .

Let  $K \geq 1$  be the integer required by Cor 7.11 of [17] for  $\varepsilon/2$  (in place of  $\varepsilon$ ),  $\mathcal{F}$  and  $T$ . By applying Cor. 7.11 of [17], one obtains (note that  $M(M_k(B \otimes \mathcal{Z}_0))$  is unital), one obtains a unitary  $v \in M_{K+1}(M(M_k(B \otimes \mathcal{Z}_0)))$  such that

$$\|u^*(d_K \circ j_{w,z}(a))u - (d_K \circ j_{w,z}(a) \oplus \varphi_{\text{odd}}(a))\| < \varepsilon \quad \text{for all } a \in \mathcal{F}.$$

□

**Lemma 9.6.** *For any  $\varepsilon > 0$ , there is  $\delta > 0$  satisfying the following: for any  $e \in A_+$  with  $\|e\| \leq 1$  and any  $a \in A$  with  $\|a\| \leq 1$ ,*

$$\|e^{1/2}ae^{1/2} - ea\| < \varepsilon$$

whenever

$$\|ea - ae\| < \delta.$$

In the following statement we keep notations in 9.4 and 9.5.

**Theorem 9.7.** *Let  $A$  be a non-unital separable amenable  $C^*$ -algebra. Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be finite subset.*

*There exists  $\delta > 0$  with  $\delta < \varepsilon/2$ , a finite subset  $\mathcal{G} \subset A$  with  $\mathcal{F} \subset \mathcal{G}$  and an integer  $K \geq 1$  satisfying the following: For any  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $\varphi : A \rightarrow M_k(\tilde{B} \otimes \mathcal{Z}_0)$  (for any non-unital separable simple  $C^*$ -algebra  $B$  with continuous scale and any integer  $k \geq 1$ ) such that if there are homomorphisms  $\psi_{z,w} : M_k(\mathcal{Z}_0) \rightarrow W$  and  $\psi_{w,z} : W \rightarrow M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \cong M_k(\mathcal{Z}_0)$  which map strictly positive elements to strictly positive elements such that*

$$\|\pi \circ (\varphi(a)) - (\psi_{w,z} \circ \psi_{z,w} \circ \pi \circ (\varphi(a)))\| < \delta \quad \text{for all } a \in \mathcal{G},$$

where  $\pi : M_k(\tilde{B} \otimes \mathcal{Z}_0) \rightarrow M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0)$  is the quotient map,

then there exists an  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map  $L_0 : A \rightarrow M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$  and an  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map  $L_1 : A \rightarrow M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$  such that

$$\|L_0(a) \oplus L_1(a) - \varphi(a) \oplus d_K \circ s \circ \varphi^\pi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F},$$

where  $\varphi^\pi = \psi_{w,z} \circ \psi_{z,w} \circ \pi \circ \varphi$ ,  $s : M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \rightarrow M_k(\tilde{B} \otimes \mathcal{Z}_0)$  is the nature embedding,

$$L_0(a) = p_m^{1/2}(\varphi(a) \oplus d_K \circ s \circ \varphi^\pi(a))p_m^{1/2} \text{ for all } a \in A$$

for some  $m \geq m_0$ , where  $\{p_m\}$  is an approximate identity for  $M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$  and, there are  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $L_{0,0} : A \rightarrow W$  and  $L_{0,0}(\mathcal{F})$ - $\varepsilon/2$ -multiplicative completely positive contractive linear map  $L_{w,b} : W \rightarrow M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$  such that  $L_1 = L_{w,b} \circ L_{0,0}$ .

*Proof.* Fix  $1/2 > \varepsilon > 0$  and a finite subset  $\mathcal{F} \subset A$ . We may assume that  $\mathcal{F} \subset A^1$ .

Let  $\mathcal{G} = \{ab : a, b \in \mathcal{F}\} \cup \mathcal{F}$ . Let  $\{e_n\} \subset M_k(B)$  be as an approximate identity as described in 9.4.

Let  $\delta_1 > 0$  (in place of  $\delta$ ) be in 9.6 for  $\varepsilon/64$ .

Let  $\delta = \min\{\delta_1/(13) \cdot 64, \varepsilon/256\}$ . We view  $M_k(\tilde{B} \otimes \mathcal{Z}_0)$  as a  $C^*$ -subalgebra of  $M(M_k(B \otimes \mathcal{Z}_0))$ . Suppose that  $\varphi : A \rightarrow M_k(\tilde{B} \otimes \mathcal{Z}_0)$  is  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map. Suppose that there are homomorphisms  $\psi_{z,w} : M_k(\mathcal{Z}_0) \rightarrow W$  and  $\psi_{w,z} : W \rightarrow M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0)$  such that

$$\|\pi \circ \varphi(a) - (\psi_{w,z} \circ \psi_{z,w} \circ \pi \circ (\varphi(a)))\| < \delta \text{ for all } a \in \mathcal{G}. \quad (\text{e } 9.10)$$

Recall that  $\varphi^\pi = \psi_{w,z} \circ \psi_{z,w} \circ \pi \circ \varphi$ . Put  $\varphi^W = \psi_{z,w} \circ \pi \circ \varphi$ . Thus  $\psi_{w,z} \circ \varphi^W = \varphi^\pi$ .

Let  $K$  be the integer in 9.5 associated with  $\delta$  (in place of  $\varepsilon$ ) and  $\varphi^W(\mathcal{G})$  (in place of  $\mathcal{F}$ ).

By applying 9.5, we obtain a unitary  $U_1 \in M_{K+2}(M(M_k(\tilde{B} \otimes \mathcal{Z}_0)))$  such that

$$\|\pi(U_1)^* \pi \circ \varphi_W(\varphi^W(a)) \pi(U_1) - \text{diag}(d_{K+1} \circ \psi_{w,z} \circ \varphi^W(a), \pi \circ \varphi_{\text{odd}}(\varphi^W(a)))\| < \delta \quad (\text{e } 9.11)$$

for all  $a \in \mathcal{G}$ .

Let  $s : M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0) \rightarrow M_k(\tilde{B} \otimes \mathcal{Z}_0)$  be the embedding such that

$$\pi \circ s(a) = a \text{ for all } a \in M_k(\mathbb{C} \cdot 1_{\tilde{B}} \otimes \mathcal{Z}_0).$$

Consider  $L_{1,1} : A \rightarrow M_k(\tilde{B} \otimes \mathcal{Z}_0)$  defined by  $L_{1,1} = \varphi_W \circ \varphi^W$  and  $L'_{1,0} : A \rightarrow M_{K+2}(M_k(\tilde{B} \otimes \mathcal{Z}_0))$  defined by

$$L'_{1,0}(a) = \text{diag}(d'_{K+1} \circ s \circ \psi_{w,z} \circ \varphi^W(a), \varphi_{\text{odd}}(\varphi^W(a))) \text{ for all } a \in A,$$

where notation  $d'_m(c)$  means the following:

$$d'_m(c) = \text{diag}(\overbrace{c, c, \dots, c}^m).$$

By 9.5, there is another unitary  $U_2 \in M_{K+2}(M(M_k(B \otimes \mathcal{Z}_0)))$  such that

$$\|U_2^* L'_{1,0}(a) U_2 - d'_{K+1} \circ s \circ \psi_{w,z} \circ \varphi^W(a)\| < \delta \quad (\text{e } 9.12)$$

for all  $a \in \mathcal{G}$ .



Define

$$L_{1,0}(a) = d'_{K+1} \circ s \circ \varphi^\pi(a) \text{ for all } a \in A.$$

Put  $\Phi = \varphi \oplus d'_K \circ s \circ \varphi^\pi$  and  $U = U_1 U_2$ . Then  $\pi \circ \Phi = \pi \circ L_{1,0}$ .

By (e 9.11), for each  $a \in \mathcal{G}$ , there exists  $b(a), b'(a) \in M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$  with  $\|b(a)\|, \|b'(a)\| \leq 1$  such that

$$\|U^* L_{1,1}(a)U - L_{1,0}(a) + b(a)\| < 2\delta \text{ and} \quad (\text{e 9.13})$$

$$\|U^* L_{1,1}(a)U - \Phi(a) + b'(a)\| < 2\delta \quad (\text{e 9.14})$$

for all  $a \in \mathcal{G}$ .

Put  $\bar{e}_n = \text{diag}(\overbrace{e_n, e_n, \dots, e_n}^{K+2})$ ,  $n = 1, 2, \dots$ . Let  $p_n = U^* \bar{e}_n U$ ,  $n = 1, 2, \dots$ . Then  $\{p_n\}$  is an approximate identity for  $M_{K+2}(M_k(B \otimes \mathcal{Z}_0))$ . Let  $S = \mathbb{N} \setminus \{4n, 4n+1 : n \in \mathbb{N}\}$ . If  $m \in S$ ,

$$(1 - p_m)(p_{4n} - p_{4n-1}) = \begin{cases} (p_{4n} - p_{4n-1}) & \text{if } m < 4n - 1; \\ 0 & \text{if } m > 4n - 1 \end{cases} \text{ and} \quad (\text{e 9.15})$$

$$p_m(1 - p_m)(p_{4n} - p_{4n-1}) = 0 \text{ for all } m. \quad (\text{e 9.16})$$

There is  $N \geq 1$  such that, for any  $m \geq N$  and  $m \in S$ ,

$$\|(1 - p_m)(U^* L_{1,1}(a)U) - (1 - p_m)\Phi(a)\| < 4\delta \text{ and} \quad (\text{e 9.17})$$

$$\|(U^* L_{1,1}(a)U)(1 - p_m) - \Phi(a)(1 - p_m)\| < 4\delta \text{ for all } a \in \mathcal{G}. \quad (\text{e 9.18})$$

Note that, by the construction of  $\varphi_W$  and (e 9.16),

$$(1 - p_m)(U^* L_{1,1}(a)U) = (U^* L_{1,1}(a)U)(1 - p_m) \quad (\text{e 9.19})$$

$$= (1 - p_m)(U^* L_{1,1}(a)U)(1 - p_m) \text{ for all } a \in A. \quad (\text{e 9.20})$$

It follows from (e 9.17) and (e 9.19)

$$\|p_m \Phi(a) - \Phi(a)p_m\| < 8\delta \text{ for all } a \in \mathcal{G}. \quad (\text{e 9.21})$$

Moreover, the map  $a \mapsto (1 - p_m)(U^* L_{1,1}(a)U)$  is a  $\mathcal{G}$ - $\delta$ -multiplicative. Define

$$L(a) = p_m \Phi(a) + (1 - p_m)(U^* L_{1,1}(a)U) \text{ for all } a \in A.$$

Then, by (e 9.17),

$$\|L(a) - \Phi(a)\| < 4\delta \text{ for all } a \in \mathcal{G}. \quad (\text{e 9.22})$$

In particular,

$$\|L(ab) - L(a)L(b)\| < 5\delta \text{ for all } a, b \in \mathcal{G}. \quad (\text{e 9.23})$$

We compute that

$$L(ab) = p_m \Phi(ab) + (1 - p_m)(U^* L_{1,1}(ab)U) \text{ for all } a, b \in A, \quad (\text{e 9.24})$$

and, for all  $a, b \in \mathcal{G}$ , by (e 9.16), (e 9.20) and (e 9.21),

$$\begin{aligned} L(a)L(b) &= (p_m \Phi(a) + (1 - p_m)(U^* L_{1,1}(a)U))(p_m \Phi(b) + (1 - p_m)(U^* L_{1,1}(b)U)) \\ &= p_m \Phi(a)p_m \Phi(b) + ((1 - p_m)(U^* L_{1,1}(a)U))(1 - p_m)(U^* L_{1,1}(b)U) \\ &\approx_{8\delta+\delta} p_m \Phi(a)\Phi(b)p_m + (1 - p_m)(U^* L_{1,1}(ab)U). \end{aligned}$$

Combining this with (e 9.24), (e 9.23)

$$\|p_m \Phi(ab) - p_m \Phi(a) \Phi(b) p_m\| < 8\delta + 5\delta = 13\delta \text{ for all } a, b \in \mathcal{G}. \quad (\text{e 9.25})$$

Therefore

$$\|p_m^{1/2} \Phi(ab) p_m^{1/2} - p_m^{1/2} \Phi(a) p_m^{1/2} p_m^{1/2} \Phi(b) p_m^{1/2}\| < 13\delta + 3\varepsilon/64 < \varepsilon/16. \quad (\text{e 9.26})$$

Define  $L_0(a) = p_m^{1/2} \Phi(a) p_m^{1/2}$  and  $L_1(a) = (1 - p_m)^{1/2} (U^* L_{1,1}(a) U) (1 - p_m)^{1/2}$ . By (e 9.22) and the choice of  $\delta_1$ , we finally have

$$\|(L_0(a) + L_1(a)) - \Phi(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}.$$

□

**Theorem 9.8.** *Let  $A$  be a non-unital separable amenable  $C^*$ -algebra which satisfies the UCT which satisfies the condition in 8.3 and let  $B$  be a separable simple  $C^*$ -algebra. For any  $\alpha \in KL(A, B)$ , there exists an asymptotic sequential morphism  $\{\varphi_n\}$  from  $A$  into  $B \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that*

$$[\{\varphi_n\}] = \alpha.$$

*Proof.* Let  $\mathcal{P} \subset \underline{K}(A)$  be a finite subset. Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset. We assume that, any  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map  $L$  from  $A$ ,  $[L]|_{\mathcal{P}}$  is well-defined.

It follows from 9.2 that there exist sequences of approximately multiplicative completely positive contractive linear maps  $\Phi_n : A \rightarrow B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  and  $\Psi_n : A \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  such that, for any finite subset  $\mathcal{Q} \subset \underline{K}(A)$ ,

$$[\Phi_n]|_{\mathcal{Q}} = \alpha|_{\mathcal{Q}} + [\Psi_n]|_{\mathcal{Q}}$$

for all sufficiently large  $n$ , where  $\Psi_n = s \circ \pi \circ \Phi_n$  (without loss of generality) and  $\pi : B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K} \rightarrow \mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  be the quotient map. Fix a sufficiently large  $n$ .

Let  $\{e_{i,j}\}$  be a system of matrix unit for  $\mathcal{K}$  and let  $E$  be the unit of the unitization of  $1_{B^\perp} \otimes \mathcal{Z}_0$ . By considering maps  $a \mapsto (E \otimes \sum_{i=1}^k e_{i,i}) \Phi_n(a) (E \otimes \sum_{i=1}^k e_{i,i})$  and maps  $a \mapsto (E \otimes \sum_{i=1}^k e_{i,i}) \Psi_n(a) (E \otimes \sum_{i=1}^k e_{i,i})$ , without loss of generality, we may assume that the image of  $\Phi_n$  is in  $M_k(B^\perp \otimes \mathcal{Z}_0)$  and that of  $\Psi_n$  is also in  $M_k(\mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0)$  for some sufficiently large  $k$ .

Define  $\iota^* : B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K} \rightarrow B^\perp \otimes \mathcal{Z}_0 \otimes \mathcal{K}$  by defining  $\iota^*(b \otimes z \otimes k) = b \otimes j^*(z) \otimes k$  for all  $b \in B^\perp$ ,  $z \in \mathcal{Z}_0$  and  $k \in \mathcal{K}$ . Note that

$$s \circ \pi(\Phi_n \oplus s \circ \pi \circ \iota^* \circ \Phi_n) = \Psi_n \oplus s \circ \pi \circ \iota^* \circ \Phi_n.$$

Let  $\delta > 0$  and let  $\mathcal{G} \subset A$  be a finite subset.

It follows from virtue of 7.13, replacing  $\Phi_n$  by  $\Phi_n \oplus s \circ \pi \circ \iota^* \circ \Phi_n$  and replacing  $\Psi_n$  by  $\Psi_n \oplus s \circ \pi \circ \iota^* \circ \Phi_n$ , and by implementing a unitary in unitization of  $M_k(\mathbb{C} \cdot 1_{B^\perp} \otimes \mathcal{Z}_0)$ , we may assume that

$$\|\pi \circ \Phi_n(g) - \varphi_{w,z} \circ \varphi_{z,w} \circ \pi(\Phi_n(a))\| < \delta \text{ for all } g \in \mathcal{G}.$$

and  $\Psi_n$  factors through  $W$ , in particular,

$$[\Psi_n]|_{\mathcal{P}} = 0. \quad (\text{e 9.27})$$

In other words,

$$[\Phi_n]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \quad (\text{e 9.28})$$

By applying 9.7, we obtains an integer  $K \geq 1$ ,  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear maps  $L_{0,n} : A \rightarrow M_k(B \otimes \mathcal{Z}_0)$ ,  $L_{1,n} : A \rightarrow M_{(K+2)k}(B^\perp \otimes \mathcal{Z}_0)$  and  $L_{2,n} : A \rightarrow M_{(K+1)k}(B^\perp \otimes \mathcal{Z}_0)$  such that

$$\|L_{0,n}(a) \oplus L_{1,n}(a) - \Phi_n(a) \oplus L_{2,n}(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}, \quad (\text{e 9.29})$$

where  $L_{1,n}$  and  $L_{2,n}$  factor through  $W$ . In particular,

$$[L_{1,n}]|_{\mathcal{P}} = [L_{2,n}]|_{\mathcal{P}} = 0. \quad (\text{e 9.30})$$

It follows that, using (e 9.28) and e 9.29

$$[L_{0,n}]|_{\mathcal{P}} = \alpha|_{\mathcal{P}}. \quad (\text{e 9.31})$$

Choose  $\varphi_n = L_{0,n}$  (for all sufficiently large  $n$ ).

□

## 10 Existence Theorem for determinant maps

**Lemma 10.1.** *Let  $A$  be a stably projectionless simple  $C^*$ -algebra such that  $Cu(A) = \text{LAff}_+(\tilde{T}(A))$  with strict comparison for positive elements and with continuous scale. Suppose  $a, b \in A \otimes \mathcal{K}_+$ . Then  $\langle a \rangle \ll \langle b \rangle$  ( $\langle a \rangle$  is compact contained in  $\langle b \rangle$ ) if and only if, there exists  $\delta > 0$ , for any  $t_0 \in T(A)$ , there exists a neighborhood  $O(t_0) \subset T(A)$  such that*

$$d_{t_0}(b) > d_\tau(a) + \delta \text{ for all } \tau \in O(t_0). \quad (\text{e 10.1})$$

*Proof.* The proof is a standard compactness argument (see, for example 5.4 of [33]). Suppose that (e 10.1) holds. Let  $f_n \in \text{LAff}_+(\tilde{A})$  such that  $f_n \nearrow \sup f_n \geq \langle b \rangle$ . Therefore, for each  $t \in T(A)$ , there exist  $n_t$  such that

$$f_{n_t}(t) > d_t(b) - \delta/8. \quad (\text{e 10.2})$$

Since each  $f_{n_t}$  is lower semi-continuous, there is a neighborhood  $U(t) \subset O(t)$  such that

$$f_{n_t}(\tau) > d_t(b) - \delta/4 \text{ for all } \tau \in U(t). \quad (\text{e 10.3})$$

It follows that

$$f_{n_t}(\tau) > d_t(b) - \delta/4 > d_\tau(a) + \delta/2 \text{ for all } \tau \in U(t). \quad (\text{e 10.4})$$

There are finitely many such  $U(t_1), U(t_2), \dots, U(t_m)$  covers  $T(A)$ . Put  $n_0 = \max\{n_{t_i} : 1 \leq i \leq m\}$ . Then, if  $\tau \in U(t_j)$ ,

$$f_{n_0}(\tau) > f_{n_{t_j}}(\tau) > d_\tau(a) + \delta/2. \quad (\text{e 10.5})$$

This implies that  $f_{n_0} > \langle a \rangle$  in  $\text{LAff}_+(\tilde{T}(A))$ .

For the converse, as in Lemma 2.2 of [4] (see Lemma 11.2 of [17]), there exists a sequence of continuous  $f_n \in \text{Aff}_+(T(A))$  such that  $f_n \nearrow b$ . Let  $g_n = f_n - \frac{1}{n}$ . Then  $g_n \nearrow b$ . The assumption that  $\langle a \rangle \ll \langle b \rangle$  implies that, for some  $n_0 \geq 1$ ,  $\langle a \rangle < g_{n_0} = f_{n_0} - \frac{1}{n_0}$  in  $Cu(A)$ . Hence

$$f_{n_0}(\tau) > d_\tau(a) + \frac{1}{n_0} \text{ for all } \tau \in T(A). \quad (\text{e 10.6})$$

Since  $f_{n_0}$  is continuous, for each  $t \in T(A)$ , there is a neighborhood  $O(t)$  such that

$$f_{n_0}(t) > d_\tau(a) + \frac{1}{2n_0} \text{ for all } \tau \in O(t). \quad (\text{e 10.7})$$

Therefore

$$d_t(b) \geq f_{n_0}(t) > d_\tau(a) + \frac{1}{2n_0} \text{ for all } \tau \in O(t). \quad (\text{e 10.8})$$

□

**Theorem 10.2.** *Let  $A$  be a stably projectionless simple exact  $C^*$ -algebra with strictly comparison for positive elements, with stable rank one and with continuous scale such that  $Cu(A) = \text{LAff}_+(\tilde{T}(A))$ . Fix  $1 > \alpha > 0$  and  $1 > \eta \geq 3/4$ . Let*

$$h_\eta \in \{f \in C([0, 1], \mathbb{R}) : f(0) = \alpha f(1)\}$$

*such that  $h_\eta$  is strictly increasing on  $[0, \eta]$ ,  $0 \leq h_\eta \leq 1$ ,  $h_\eta(0) = 0 = h_\eta(1)$ , and  $h_\eta(\eta) = 1$ .*

*Let  $c \in A_+$  with  $\|c\| = 1$  and  $b \in \overline{cAc}_+$  with  $\|b\| = 1$ . Suppose that there is a non-zero homomorphism  $\varphi : R(\alpha, 1) \rightarrow \overline{cAc}$ .*

*Then, for any  $\varepsilon > 0$ , there exists a homomorphism  $\psi : R(\alpha, 1) \rightarrow B := \overline{cAc}$  such that*

$$\sup\{|\tau(\psi(h_\eta)) - \tau(b)| : \tau \in T(A)\} < \varepsilon.$$

*Proof.* Let  $\varepsilon > 0$ . Then, since  $A$  is stably projectionless, we may assume that  $\text{sp}(b) = [0, 1]$ .

Note that  $(h_\eta)|_{[0, \eta]} : [0, \eta] \rightarrow [0, 1]$  is a bijection. Define  $h_\eta^{-1} : [0, 1] \rightarrow [0, \eta]$  to be the inverse of  $(h_\eta)|_{[0, \eta]}$ . Note that  $h_\eta \circ h_\eta^{-1} = \text{id}_{[0, 1]}$ . For each  $f \in C([0, 1], \mathbb{R})_+$ , define  $\gamma(f)(\tau) = \tau(f \circ h_\eta^{-1}(b))$  for all  $\tau \in T(A)$ .

This gives an affine continuous map from  $C([0, 1], \mathbb{R}) \rightarrow \text{Aff}(T(A))$ . Note that  $\text{Aff}(\tilde{T}(R(\alpha, 1)))$  and  $\text{LAff}(\tilde{T}(R(\alpha, 1)))_+$  are identified with

$$\begin{aligned} \{(f, r) \in C([0, 1], \mathbb{R}) \oplus \mathbb{R} : f(0) = r\alpha \text{ and } f(1) = r\} &= \{f \in C([0, 1], \mathbb{R}) : f(0) = \alpha f(1)\} \\ \text{and } LSC([0, 1], \mathbb{R}_+^\sim) \oplus_\alpha \mathbb{R}_+^\sim & \end{aligned}$$

(see 3.10), respectively. Let  $\gamma_1 = \gamma|_{\text{Aff}(\tilde{T}(R(\alpha, 1)))_+}$ . Then

$$\gamma_1(\hat{h}_\eta)(\tau) = \tau(h_\eta \circ h_\eta^{-1}(b)) = \tau(b). \quad (\text{e 10.9})$$

It induces an ordered semi-group homomorphism  $\gamma_1 : \text{LAff}(\tilde{T}(R(\alpha, 1)))_+ \rightarrow \text{LAff}(\tilde{T}(A))_+$ . Note  $\gamma_1$  takes continuous functions to continuous functions. Let  $r : Cu(R(\alpha, 1)) \rightarrow \text{LAff}(\tilde{T}(R(\alpha, 1)))_+$  be the rank function defined in 3.10. Define an order semi-group homomorphism  $\gamma_2 : Cu(R(\alpha, 1)) \rightarrow \text{LAff}_+(\tilde{T}(A))$  by

$$\gamma_2(\langle(f, r)\rangle) = (1 - \varepsilon/4)\gamma_1(r(\langle(f, r)\rangle)) + (\varepsilon/4)Cu(\varphi)(\langle(f, r)\rangle). \quad (\text{e 10.10})$$

We verify that  $\gamma_2$  is a morphism in **Cu**. Since the rank function preserves the suprema of increasing sequences, it is easy to check that  $\gamma_2$  also preserves the suprema of increasing sequences. Suppose that  $\langle f \rangle \ll \langle g \rangle$  in  $Cu(R(\alpha, 1))$ . There is a sequence of  $c_n \in Cu(R(\alpha, 1))$  such that  $r(c_n)$  is continuous and  $r(c_n) \nearrow r(\langle(g, s)\rangle)$  (see 3.10). Note that  $c_n$  can be identified with an element in  $LSC([0, 1], (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+) \oplus_\alpha (\mathbb{R}^\sim \setminus \{0\} \sqcup \mathbb{Q})_+$ , at each point  $t$ , we identify  $r(c_n)(t)$  with the corresponding values of  $c_n(t)$  in  $\mathbb{R}_+^\sim$ —that is,  $[r] \in \mathbb{Q}_+$  is regarded as  $r \in \mathbb{R}_+$ .

Put  $c = \sup_n r(c_n)$ . Then  $r(c) = r(\langle g \rangle)$ . For any  $\varepsilon_1 > 0$ ,  $(1 + \varepsilon_1)r(c) \geq \langle g \rangle$  in  $Cu(R(\alpha, 1))$ . Since  $\langle f \rangle \ll \langle g \rangle$ , there exists  $n_0 \geq 1$  such that

$$(1 + \varepsilon_1)r(c_{n_0}) \geq \langle f \rangle. \quad (\text{e 10.11})$$

This, in particular, implies that  $r(\langle f \rangle)$  is a bounded function. Let  $M > 0$  such that  $M = \sup\{d_\tau(f) : \tau \in T(A)\}$ . It follows that  $(1 + \varepsilon_1)r(c_{n_0}) \geq r(\langle (f, b) \rangle)$ .

Now let  $z_n \in Cu(R(\alpha, 1))$  such that  $z_n \nearrow \sup z_n \geq \gamma_2(\langle g, s \rangle)$ . By 10.1, there exists  $\delta > 0$  such that, for each  $t \in T(A)$ , there is a neighborhood  $U(t)$  such that

$$d_t(\varphi(g)) > d_\tau(\varphi(f)) + \delta \text{ for all } \tau \in U(t). \quad (\text{e 10.12})$$

Choose  $0 < \varepsilon_1 < \varepsilon \cdot \delta/16(M + 1)$ . Then, for some  $n_0 \geq 1$ ,

$$(1 - \varepsilon/4)(1 + \varepsilon_1)\gamma_1(r(c_{n_0})) > (1 - \varepsilon/4)(1 + \varepsilon_1)\gamma_1(r(\langle f \rangle)). \quad (\text{e 10.13})$$

Since  $r(c_{n_0})$  is continuous,  $\gamma_1(r(c_{n_0}))$  is also continuous. Therefore, for each  $t \in T(A)$ , there is a neighborhood  $O(t)$  such that

$$(1 - \varepsilon/4)\gamma_1(r(c_{n_0}))(t) > (1 - \varepsilon/4)\gamma_1(r(\langle f \rangle))(\tau) - \varepsilon_1 \text{ for all } \tau \in O(t). \quad (\text{e 10.14})$$

Put  $N(t) = O(t) \cap U(t)$ . Then, by (e 10.12) and (e 10.14) as well as (e 10.10),

$$\gamma_2(\langle g \rangle)(t) > \gamma_2(r(\langle f \rangle))(\tau) + \varepsilon\delta/2 \text{ for all } \tau \in N(t). \quad (\text{e 10.15})$$

It follows from 10.1 that  $\gamma_2(\langle f \rangle) \ll \gamma_2(\langle g \rangle)$ . This shows that  $\gamma_2$  is a morphism in **Cu**. Since  $K_0(R(\alpha, 1)) = \{0\}$ , it induces a morphism  $\gamma_2^\sim : Cu^\sim(R(\alpha, 1)) \rightarrow Cu^\sim(A)$ .

It follows from [43] that there exists a homomorphism  $\psi : R(\alpha, 1) \rightarrow B = \overline{cAc}$  such that

$$d_\tau(\psi(g)) = \gamma_2(\langle g \rangle)(\tau) \text{ for all } \tau \in T(A) \quad (\text{e 10.16})$$

and for all  $g \in R(\alpha, 1)_+$ . There is  $f \in R(\alpha, 1)_+$  such that  $d_\tau(f) = \tau(h_\eta)$  for all  $\tau \in T(R(\alpha, 1))$  (see 3.7). Therefore

$$d_\tau(\psi(f)) = \lim_{n \rightarrow \infty} \tau(\psi(f^{1/n})) = \lim_{n \rightarrow \infty} \tau \circ \psi(f^{1/n}) \quad (\text{e 10.17})$$

$$= d_{\tau \circ \psi}(f) = (\tau \circ \psi)(h_\eta) \text{ for all } \tau \in T(A). \quad (\text{e 10.18})$$

Then, by (e 10.10) and (e 10.16),

$$|d_t(\psi(f)) - \gamma_1(r(f))(t)| < \varepsilon/4 \text{ for all } t \in T(R(\alpha, 1)). \quad (\text{e 10.19})$$

Since  $\gamma_1(r(f)) = \gamma_1(\hat{h}_\eta)$ , we estimate that

$$\sup\{|\tau \circ \psi(h_\eta) - \tau(b)| : \tau \in T(A)\} < \varepsilon.$$

The lemma follows.  $\square$

**10.3.** Write  $A$  be the AH-algebras of real rank zero with unique tracial state as associated with  $B_T$  in section 6. So  $B_T = \lim_{n \rightarrow \infty} (B_n, \Phi_n)$ . Write

$$B_n = W_n \oplus E_n \text{ and } E_n = M_{(n!)^2}(A(W, \alpha_n)), \quad n = 1, 2, \dots$$

We may write  $A = \overline{\bigcup_{n=1}^\infty C_n}$ , where  $C_n = C_{n,1} \oplus C_{n,2}$ ,  $C_{n,1} \oplus C_{n,2} \subset C_{n+1,1} \oplus C_{n+1,2}$  and  $C_{n,1}$  is a circle algebra and  $C_{n,2}$  is a homogeneous  $C^*$ -algebra with torsion  $K_1$ . In fact,  $C_{n,1}$

may be written as  $M_{r(n)}(C(X_n))$ , where  $X_n$  is a finite CW complex with dimension no more than 3 and  $r(n) \geq 6$  (see [12]). In particular (by [42]),  $K_1(C_{n,2}) = U(C_{n,2})/U_0(C_{n,2})$ . We use  $j_n : C_n \rightarrow C_{n+1}$  for the embedding.

Fix a finitely generated subgroup  $F_0 \subset K_1(B_T)$ . We may assume that  $F'_0 \subset K_1(B_n)$  such that  $(\Phi_{n,\infty})_{*1}(F'_0) = F_0$ . Write  $B_n = E_n \oplus W_n$ , where  $E_n = M_{(n!)^2}(A(W, \alpha_n))$ . We also write

$$C_{k,1} = M_{r(k(1))}(C(\mathbb{T})) \oplus M_{r(k(2))}(C(\mathbb{T})) \oplus \cdots M_{r(k(m_f))}(C(\mathbb{T})).$$

with the identity of each summand being  $p_j$ ,  $j = 1, 2, \dots, k(m_f) = m_f$ ,—here we denote  $m_f$  by  $k(m_f)$  to emphasis that it is correspond to  $C_k$ . We choose  $n \geq 1$  so that  $n \geq m_f$ . Put  $F'_1 = \pi'_{n*1}(F'_0)$ , where  $\pi'_n : B_n \rightarrow A$  defined by  $\pi'_n(a \oplus b) = \pi(a)$  for all  $a \in A(W, \alpha_n)$  and  $b \in W_n$ , where  $\pi_n : A(W, \alpha_n) \rightarrow M_{(n!)^2}(A)$  is the quotient map. Note that  $\pi_{n*1} : K_1(B_n) \rightarrow K_1(A)$  is an isomorphism. We may assume that  $F'_1 \subset (j_{k,\infty})_{*1}(K_1(C_k))$ . Let  $\tilde{F} = \pi_{n*1}^{-1}((j_{k,\infty})_{*1}(K_1(C_k)))$  and  $F = (\Phi_{n,\infty})_{*1}(\tilde{F})$ . (Here, we identify  $K_1(M_{(n!)^2}(A))$  with  $K_1(A)$  and  $K_1(M_{(n!)^2}(C_k))$  with  $K_1(C_k)$ .)

The subgroup  $F$  may be called *the standard subgroup* of  $K_1(B_T)$ .

In what follows  $\text{tr}$  is the unique tracial state on  $Q$ . We will define an injective homomorphism  $j_{F,u} : F \rightarrow U(B_T)/CU(B_T)$ . We identify  $\widetilde{A(W, \alpha_n)}$  with the following  $C^*$ -algebra:

$$\{(f_\lambda, a) \in C([0, 1], Q \otimes Q) \oplus A : f_\lambda(0) = (s(a - \lambda) \otimes e_{\alpha_n}) + \lambda \cdot 1_{Q \otimes Q} \text{ and } f_\lambda(1) = s(a - \lambda) \otimes 1_Q + \lambda \cdot 1_{Q \otimes Q}\},$$

where  $\lambda \in \mathbb{C}$  and  $a - \lambda = a - \lambda \cdot 1_A \in A$ . Note that  $(f, 1_A)$ , where  $f(t) = 1_Q \otimes 1_Q$ , is added to  $A(W, \alpha)$ .

Write  $F = \mathbb{Z}^{k(m_f)} \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \cdots \mathbb{Z}/k_{m_t}\mathbb{Z}$ . Put  $m = k(m_f) + k(m_t)$ . Let  $x_1, x_2, \dots, x_{k(m_f)}$  be the free cyclic generators for  $\mathbb{Z}^{k(m_f)}$  and  $x_{0,j}$  be cyclic generators for each  $\mathbb{Z}/k_j\mathbb{Z}$ ,  $j = 1, 2, \dots, k(m_t)$ , respectively.

Fix unitaries  $z'_1, z'_2, \dots, z'_{k(m_f)}, z'_{0,1}, z'_{0,2}, \dots, z'_{0,k(m_t)} \in C_{k,1}$  such that  $[z'_i] = x_i$ ,  $i = 1, 2, \dots, k(m_f)$  and  $[z'_{0,j}] = x_{0,j}$ ,  $j = 1, 2, \dots, k(m_t) = m_t$ . Note that  $(z'_{0,j})^{k_j} \in U_0(C_{n,2})$ . We may choose  $z'_{0,j}$  so that  $(z'_{0,j})^{k_j} \in CU(C_{n,2})$ . We further assume that  $z'_j = \text{diag}(z_j^{(0)}, 1, \dots, 1)$ , where  $z_j^{(0)}$  is the standard unitary generator for  $C(\mathbb{T})$ ,  $j = 1, 2, \dots, k(m_f)$ .

We write  $s(z'_j) = \exp(ih'_{j,0}) \exp(ih'_{j,1})$ , where  $h'_{j,0}, h'_{j,1} \in s(p_j)Q_{s.a.}s(p_j)$  (Note that here we use the fact that the exponential rank for  $A$  is  $1 + \varepsilon$  (see [26])). Let  $h''_{j,0}, h''_{j,1} \in \mathbb{R}$  such that  $h''_{j,l} = \text{tr}(h'_{j,l})$ ,  $l = 0, 1$ . Put  $z_j = z'_j \exp(-2i\pi h''_{j,1}) \exp(-2i\pi h''_{j,0})$ ,  $j = 1, 2, \dots, m(k)$ . Then  $[z_j] = [z'_j] = x_j$ . Note that  $s(z_j) = \exp(2i\pi h_{j,0}) \exp(2i\pi h_{j,1})$  such that  $h_{j,0}, h_{j,1} \in (s(p_j)Qs(p_j))_{s.a.}$  and  $\text{tr}(h_{j,0}) + \text{tr}(h_{j,1}) = 0$ ,  $j = 1, 2, \dots, k(m_f)$ . We also choose  $z_{0,j}$  and  $s(z_{0,j}) = \exp(ih_{j,0,0}) \exp(ih_{j,0,1})$  such that  $\text{tr}(h_{j,0,0}) + \text{tr}(h_{j,0,1}) = 0$ , and  $[z_{0,j}] = x_{0,j}$ .

Define  $u_j = (f_j, z_j)$  as follows.

$$f_j(t) = (s(z_j) \otimes e_{\alpha_n}) \oplus ((\exp(i2t\pi h_{j,0}) \exp(i2t\pi h_{j,1})) \otimes (1 - e_{\alpha_n})) \text{ for all } t \in [0, 1]. \quad (\text{e } 10.20)$$

Note that

$$f_j(0) = (s(z_j) \otimes e_{\alpha_n}) \oplus (1 \otimes (1_Q - e_{\alpha_n})) \text{ and} \quad (\text{e } 10.21)$$

$$f_j(1) = (s(z_j) \otimes e_{\alpha_n}) \oplus (\exp(i2\pi h_{j,0}) \exp(i2\pi h_{j,1}) \otimes (1 - e_{\alpha_n})) = s(z_j) \otimes 1_Q. \quad (\text{e } 10.22)$$

In fact

$$f_j(t) = \exp(2i\pi d_{j,0}(t)) \exp(2i\pi d_{j,1}(t)), \quad (\text{e } 10.23)$$

where

$$d_{j,0}(t) = h_{j,0} \otimes e_{\alpha_n} + th_{j,0} \otimes (1_Q - e_{\alpha_n}) \text{ and} \quad (\text{e } 10.24)$$

$$d_{j,1}(t) = h_{j,1} \otimes e_{\alpha_n} + th_{j,1} \otimes (1_Q - e_{\alpha_n}). \quad (\text{e } 10.25)$$

In particular,  $(f_j, z_j) \in A(\widetilde{W}, \alpha_n)$  and  $u_j \in U(A(\widetilde{W}, \alpha_n))$ ,  $j = 1, 2, \dots, k(m_f)$ .

Write  $u_j = \zeta_j + \mu(u_j)$ , where  $\zeta_j \in A(\widetilde{W}, \alpha_n)$  and  $\mu(u_j)$  is a scalar. Since  $d_{j,0}, d_{j,1} \in A(W, \alpha_n)_{s.a.}$ ,  $\mu(u_j) = 1$ . In particular,  $(f, z_j) \in A(\widetilde{W}, \alpha_n)$  and  $u_j \in U(A(\widetilde{W}, \alpha_n))$ ,  $j = 1, 2, \dots, k(m_f)$ .

Let  $u_{0,j} = (f_{0,j}, z_{0,j}) \in A(\widetilde{W}, \alpha_n)$  be defined as follows:

$$f_{0,j}(t) = \exp(2i\pi d_{j,0,0}(t)) \exp(2i\pi d_{j,0,1}(t)), \quad (\text{e } 10.26)$$

where

$$d_{j,0,0}(t) = h_{j,0,0} \otimes e_{\alpha_n} + th_{j,0,0} \otimes (1_Q - e_{\alpha_n}) \quad \text{and} \quad (\text{e } 10.27)$$

$$d_{j,0,1}(t) = h_{j,0,1} \otimes e_{\alpha_n} + th_{j,0,1} \otimes (1_Q - e_{\alpha_n}). \quad (\text{e } 10.28)$$

One has, for some  $\zeta_{0,j} \in A(W, \alpha_n)$ ,

$$u_{0,j} = \zeta_{0,j} + 1_{A(\widetilde{W}, \alpha_n)}.$$

The map  $J_{u,F,n} : \tilde{F} \rightarrow U(\widetilde{B_n})/CU(\widetilde{B_n})$  defined by  $x_j \mapsto \overline{u_j}$  and  $x_{0,j} \mapsto \overline{u_{0,j}}$  is an injective homomorphism and define  $J_{u,F} : F \rightarrow U(\widetilde{B_T})/CU(\widetilde{B_T})$  by identifying  $\overline{u_j}$  with  $\overline{\Phi_{n,\infty}(u_j)}$  and  $\overline{u_{0,j}}$  with  $\overline{\Phi_{n,\infty}(u_{0,j})}$ . It should be noted, by our choice,  $k_j \overline{u_{0,j}} = 0$ .

**10.4.** We keep notation used in 10.3. Define

$$E_{n,k} = \{(f, a) \in M_{(n!)^2}(C([0, 1], Q \otimes Q) \oplus M_{(n!)^2}(C_k) : f(0) = s(a) \otimes e_{\alpha_n} \text{ and } f(1) \in s(a) \otimes 1_Q\},$$

$n = 1, 2, \dots$ . Fix  $\varepsilon > 0$  and a finite subset  $\mathcal{F} \subset B_T$ . Without loss of generality, we may assume that  $\mathcal{F} \subset B_n$ . Denote by  $\mathcal{F}^{Aw} = q_{E_n}(\mathcal{F})$ , where  $q_{E_n} : B_n \rightarrow E_n = M_{(n!)^2}(A(W, \alpha_n))$  is the projection map. Let

$$C_{k,1} = \bigoplus_{i=1}^{k(m_f)} M_{r(k(i))}(C(\mathbb{T})).$$

Now write  $u_1, u_2, \dots, u_{k(m_f)} \in \tilde{E}_n$  which represent the free generators of  $K_1(E_{n,k})$ . We may assume that  $\pi_n(u_j) = z_j$ , the unitary generator for  $M_{r(k(j))}(C(\mathbb{T}))$ ,  $j = 1, 2, \dots, k(m_f)$ , and where  $\pi_n : E_n \rightarrow M_{(n!)^2}(A)$  is the quotient map. We also assume that  $z_j$  and  $u_j$  have the form (e 10.20).

Fix  $\varepsilon/2 > \delta > 0$  and a finite subset  $\mathcal{G}' \subset C_k$  with  $\mathcal{G}' \supset \pi_{n,k}(\mathcal{F}^{Aw})$ , where  $\pi_{n,k} : E_{n,k} \rightarrow M_{(n!)^2}(C_k)$  is the quotient map. Choose a finite subset  $\mathcal{F}_1 \supset \mathcal{F}^{Aw}$  such that  $\pi_{n,k}(\mathcal{F}_1) \supset \mathcal{G}'$ .

We also assume that there is an  $\mathcal{G}'$ - $\delta$ -multiplicative completely positive contractive linear map  $L : A \rightarrow C_k$  such that

$$\|L(a) - a\| < \delta/4 \text{ for all } a \in \mathcal{G}', \quad (\text{e } 10.29)$$

where we also use  $L$  to denote  $L \otimes \text{id}_{M_{(n!)^2}} : M_{(n!)^2}(A) \rightarrow M_{(n!)^2}(C_k)$ . Choose  $\delta > \delta_0 > 0$  such that, for any  $(f, a) \in \mathcal{F}_1$ , if  $|t - t'| < 2\delta_0$ ,

$$\|f(t) - f(t')\| < \delta/16 \text{ for all } t, t' \in [0, 1]. \quad (\text{e } 10.30)$$

Define  $\tilde{L} : E_n \rightarrow E_{n,k}$  as follows:  $\tilde{L}((f, a)) = (g, L(a))$ , where

$$g(t) = \begin{cases} ((1 - 2t/\delta_0)s(L(a)) \otimes e_{\alpha_n} + \frac{2t}{\delta_0}s(a) \otimes e_{\alpha_n}) & \text{for all } t \in [0, \delta_0/2], \\ f(\frac{t - \delta_0/2}{1 - \delta_0/2}) & \text{for all } t \in (\delta_0/2, 1 - \delta_0/2], \\ \frac{1-t}{\delta_0/2}s(a) + \frac{1-t+\delta_0/2}{\delta_0/2}s(L(a)) \otimes 1_Q & \text{for all } t \in (1 - \delta_0/2, 1]. \end{cases}$$



One verifies that  $\tilde{L}$  is an  $\mathcal{F}_1$ - $\delta/2$ -multiplicative completely positive contractive linear map from  $E_n$  into  $E_{n,k}$ .

We now assume that  $\alpha_n < \alpha_{n+1}$ . Let  $r_1 = \frac{1-\alpha_{n+1}}{1-\alpha_n}$  and  $r_2 = \frac{\alpha_{n+1}-\alpha_n}{1-\alpha_n}$ . Let  $1 > \eta > 3/4$  and  $\mu_j \geq 0$ ,  $j = 1, 2, \dots, k(m_f)$ . Let  $\omega_j = \mu_j / \text{tr}(s(p_j))$ ,  $j = 1, 2, \dots, k(m_f)$ .

Fix a continuous increasing surjective function  $g_1 : [0, \eta] \rightarrow [0, 1]$  such that  $g_1(0) = 0$ ,  $g_1(\eta) = 1$  and decreasing surjective function  $g_2 : [\eta, 1] \rightarrow [0, 1]$  such that  $g_2(\eta) = 1$ ,  $g_2(1) = 0$ . Define  $h|_{[0, \eta]} = g_1$  and  $h|_{[\eta, 1]} = g_2$ . In particular,  $h(0) = 0$  and  $h(1) = 0$ .

Define a unital homomorphism  $\varphi_{c,R}^f : M_{(n!)^2}(C_{k,1}) \rightarrow M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$  such that

$$\varphi_{c,R}^f(z_j)(t) = s(z_j) \exp(i2\pi(\omega_j/r_2)h(t))s(p_j) \otimes e_{r_2} \text{ for all } t \in [0, 1]. \quad (\text{e } 10.31)$$

Define  $\varphi_{c,R} = \varphi_{c,R}^f|_{M_{(n!)^2}(C_{k,1})} \oplus (\varphi_{c,R}^t|_{M_{(n!)^2}(C_{k,2})}) : M_{(n!)^2}(C_k) \rightarrow M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$ , where

$$\varphi_{c,R}^t(a)(t) = s(a) \otimes e_{r_2} \text{ for all } t \in [0, 1]. \quad (\text{e } 10.32)$$

Let  $\varphi_{A,R} : E_n \rightarrow M_{(n!)^2}(C([0, 1], Q) \otimes e_{r_2})$  by  $\varphi_{A,R}((f, a)) = \psi_{c,R} \circ L \circ \pi_A(a)$  for all  $a \in E_n$  and where  $\pi_A : M_{(n!)^2}(A(W, \alpha_n)) \rightarrow M_{(n!)^2}(A)$  is the quotient map.

Now define a completely positive contractive linear map  $\Psi : E_n \rightarrow M_{(n!)^2}(A(W, \alpha_{n+1}))$  defined as follows. We will use some of the notation in section 6. Define (see section 6 for the notation)

$$P_a(\Psi((f, a))) = L(a) \text{ and} \quad (\text{e } 10.33)$$

$$\begin{aligned} P_f(\Psi((f, a))) &= \text{diag}(P_f \circ \varphi_{R,r_1} \circ \varphi_{A,R,\alpha_n}(\tilde{L}(f, a)), (\varphi_{A,R}(f, a))) \\ &= \text{diag}(P_f \circ \varphi_{R,r_1} \circ \varphi_{A,R,\alpha_n}(g, L(a)), (\varphi_{A,R}(f, a))). \end{aligned} \quad (\text{e } 10.34)$$

Note that

$$P_f(\Psi(f, a))(0) = \text{diag}(s(L(a)) \otimes e_{\alpha_n r_1}, s(L(a)) \otimes e_{r_2}) = s(L(a)) \otimes e_{\alpha_n r_1} \text{ and} \quad (\text{e } 10.35)$$

$$P_f(\Psi(f, a))(1) = \text{diag}(s(L(a)) \otimes e_{r_1}, s(L(a)) \otimes e_{r_2}) = s(L(a)) \otimes 1_Q. \quad (\text{e } 10.36)$$

Let

$$\begin{aligned} W_j(t) &= (\exp(i2\pi h_{j,0}) \exp(i2\pi h_{j,1}) \otimes e_{\alpha_n r_1} \oplus (\exp(i2\pi t h_{j,0}) \exp(i2\pi t h_{j,1}) \otimes (e_{r_1} - e_{\alpha_n r_1})) \\ &\quad + s(z_j) \exp(i2\pi(\omega_j/r_2)h(t))s(p_j) \otimes e_{r_2}, \quad j = 1, 2, \dots, k(m_f). \end{aligned}$$

Let  $E'_{n+1} := M_{(n!)^2}(A(W, \alpha_{n+1}))$ , then in  $\tilde{E}'_{n+1}$  (with large  $\mathcal{G}'$ ),

$$\|\Psi(u_j) - (W_j, z_j)\| < \delta, \quad j = 1, 2, \dots, k(m_f). \quad (\text{e } 10.37)$$

(Here the unitalization of  $\Psi$  is also denoted by  $\Psi$ .) Therefore there exists  $H_{j,00} \in (E'_{n+1})_{s.a.}$  with  $\|H_{j,00}\| \leq 2 \arcsin(\delta/2)$  such that

$$[\Psi(u_j)] = \exp(i2\pi H_{j,00})(W_j, z_j), \quad j = 1, 2, \dots, k(m_f). \quad (\text{e } 10.38)$$

Put

$$H_{j,0}(t) = h_{j,0} \otimes (e_{\alpha_n r_1} \oplus e_{r_2}) \oplus t h_{j,0} \otimes (e_{r_1} - e_{\alpha_n r_1}), \quad (\text{e } 10.39)$$

$$H_{j,1}(t) = h_{j,1} \otimes (e_{\alpha_n r_1} \oplus e_{r_2}) \oplus t h_{j,1} \otimes (e_{r_1} - e_{\alpha_n r_1}) \text{ and} \quad (\text{e } 10.40)$$

$$H_{j,2}(t) = (\omega_j/r_2)h(t)s(p_j) \otimes e_{r_2}. \quad (\text{e } 10.41)$$

Noting  $h(0) = 0$  and  $h(1) = 0$ , we see that  $H_{j,l}(t) \in M_{(n!)^2}(R(\alpha_{n+1}, 1))$ . Therefore

$$\varphi_{A,R,\alpha_{n+1}}([\Psi(u_j)]) = \exp(i2\pi H_{j,00}) \exp(i2\pi H_{j,0}) \exp(i2\pi H_{j,1}) \exp(i2\pi H_{j,2}). \quad (\text{e } 10.42)$$

We compute that, for all  $t \in [0, 1]$ ,

$$\text{tr}(H_{j,00} + H_{j,0} + H_{j,1} + H_{j,2})(t) = \text{tr}(H_{j,00}) + (\omega_j/r_2)h(t) \cdot \text{tr}(s(p_j))\text{tr}(e_{r_2}) \quad (\text{e 10.43})$$

$$= \text{tr}(H_{j,00}) + \mu_j h(t). \quad (\text{e 10.44})$$

It follows that, in  $E''_{n+1} = M_{(n!)^2}(R(\alpha_{n+1}, 1))$ , for all  $t \in [0, 1]$ ,

$$|D_{E''_{n+1}}(\varphi_{A,R,\alpha_{n+1}}(\lceil \Psi(u_j) \rceil))(t) - \mu_j h(t)| < \delta. \quad (\text{e 10.45})$$

Let

$$\begin{aligned} W_{0,j}(t) &= (\exp(i2\pi h_{j,0,0}) \exp(i2\pi h_{j,0,1}) \otimes e_{\alpha_n r_1} \oplus (\exp(i2\pi t h_{j,0,0}) \exp(i2\pi t h_{j,0,1}) \otimes (e_{r_1} - e_{\alpha_n r_1})) \\ &\quad + s(z_j)s(p_j) \otimes e_{r_2}, \quad j = 1, 2, \dots, m_t. \end{aligned}$$

A similar computation shows that

$$|D_{E''_{n+1}}(\varphi_{A,R,\alpha_{n+1}}(\lceil \Psi(u_{0,j}) \rceil))(t)| < \delta. \quad (\text{e 10.46})$$

We will keep notations in 10.3 and 10.4 in the following statement.

**Lemma 10.5.** *Let  $C$  be a non-unital separable simple  $C^*$ -algebra in  $\mathcal{D}$  with continuous scale such that  $\ker \rho_C = K_0(C)$  and let  $B = B_T$  be as constructed in 6.2.*

*Let  $\varepsilon > 0$ ,  $\mathcal{F} \subset B$  be a finite subset, let  $\mathcal{P} \subset \underline{K}(B)$  be a finite subset and let  $1/2 > \delta_0 > 0$ .*

*For any finitely generated standard subgroup  $F$  (see 10.3), any finite subset  $S \subset F$ , there exists an integer  $n \geq 1$  with the following property:*

*for any finite subset  $\mathcal{U} \subset U(\tilde{B}_T)$  such that  $\overline{\mathcal{U}} \subset J_{F,u}(F) \subset J_{F,u}((\Phi_{n,\infty})_{*1}(K_1(E_n)))$  and  $\Pi(\overline{\mathcal{U}}) = S$ , where  $\Pi : U(\tilde{B})/CU(\tilde{B}) \rightarrow K_1(B)$  is the quotient map, for any homomorphism*

*$\gamma : J_{F,u}((\Phi_{n,\infty})_{*1}(K_1(E_n))) \rightarrow \text{Aff}(T(\tilde{C}))/\mathbb{Z}$ , such that  $\gamma|_{\text{Tor}(J_{u,F}((\Phi_{n,\infty})_{*1}(K_1(E_n))))} = 0$  and any  $c \in C_+$  with  $\|c\| = 1$ , there exists  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map  $\Phi : B_T \rightarrow \overline{cCc}$  such that, in  $U(\tilde{C})/CU(\tilde{C}) \cong \text{Aff}(T(\tilde{C}))/\mathbb{Z}$ ,*

$$[\Phi]|_{\mathcal{P}} = 0 \text{ and } \text{dist}(\Phi^\dagger(\bar{z}), \gamma(\bar{z})) < \delta_0 \text{ for all } z \in \mathcal{U}. \quad (\text{e 10.47})$$

*Proof.* Fix  $\varepsilon > 0$ ,  $\mathcal{F}$  and  $\mathcal{P}$  as described by this lemma. Fix  $\delta_1 > 0$ , a finite subset  $\mathcal{G} \subset B_T$ . We assume that  $\mathcal{F} \subset \mathcal{G}$ . Choose  $n_0 \geq 1$  such that there exists finite subset  $\mathcal{G}' \subset B_{n_0}$  such that, for any  $b \in \mathcal{G}$ , there exists  $b' \in \mathcal{G}'$  such that

$$\|b - \Phi_{n,\infty}(b')\| < \delta_1/64. \quad (\text{e 10.48})$$

We assume that  $\delta_1 < \min\{\delta_0/4, \varepsilon/16\}$ .

Choose  $k \geq 1$  as in 10.3 and write  $F = \mathbb{Z}^{m_f} \oplus \mathbb{Z}/k_1\mathbb{Z} \oplus \dots \oplus \mathbb{Z}/k_{m_t}\mathbb{Z}$ . Fix a set of generator  $S$  of  $F$ . Without loss of generality, we always assume that  $S$  is the set of generators of  $F$ .

To simplify notation, without loss of generality, we may assume that  $\mathcal{G} \subset \Phi_{n_0,\infty}(\mathcal{G}')$ . We also assume, without loss of generality, that  $\mathcal{P} \subset [\Phi_{n_0,\infty}](B_n)$ . Let  $\mathcal{P}' \subset \underline{K}(B_{n_0})$  be a finite subset such that  $\mathcal{P} \subset [\Phi_{n_0,\infty}](\mathcal{P}')$ .

We also assume that there exists  $L : B_T \rightarrow B_n$  such that, for all  $n \geq n_0$ ,

$$\|L(\Phi_{n,\infty}(b')) - b'\| < \delta_1/64 \text{ for all } b' \in \mathcal{G}' \quad (\text{e 10.49})$$

We further assume that  $\delta_1$  is sufficiently small and  $\mathcal{G}$  is sufficiently large so that  $[L']|_{\mathcal{P}}$  is well defined for any  $\mathcal{G}$ - $\delta_1/16$ -multiplicative completely positive contractive linear map from  $B$ .

Choose  $\delta = \frac{\delta_1}{(m_f + m_t + 2)}$  and choose  $n \geq n_0 + m_f + m_t + 2$  as in 10.4 associated with  $\delta/64$  (in place  $\varepsilon$ ) and  $\mathcal{G}$  (in place of  $\mathcal{F}$ ).

Let  $\overline{U} \subset J_{F,u}(F)$  and let  $z_j$  and  $u_j$ ,  $j = 1, 2, \dots, m_f$  and  $z_{0,j}$  and  $u_{0,j}$ ,  $j = 1, 2, \dots, m_t$  be as described in 10.3. Without loss of generality, we may assume that  $\overline{U} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{m_f}, \bar{u}_{0,1}, \dots, \bar{u}_{0,m_t}\}$ .

Choose non-zero elements  $c_{i,l} \in \overline{cC}c_+$  which are mutually orthogonal,  $i = 1, 2, \dots, m_f$ ,  $l = 1, 2$ .  
Choose  $1 > \eta_0 > 0$  such that

$$\eta_0 \leq \inf\{d_\tau(c_{j,l}) : \tau \in T(C)\}$$

for all  $1 \leq j \leq m_f$  and  $l \in \{1, 2\}$ .

Choose  $g_{j,+}, g_{j,-} \in \text{Aff}(T(C))_+$  and  $\lambda_{j,+}, \lambda_{j,-} \in \mathbb{R}_+$  such that

$$0 < g_{j,+}(\tau) \leq \eta_0, 0 < g_{j,-}(\tau) \leq \eta_0 \text{ for all } \tau \in T(C) \text{ and} \quad (\text{e 10.50})$$

$$\gamma(\bar{u}_j) = \lambda_{j,+}g_{j,+} - \lambda_{j,-}g_{j,-}, \quad j = 1, 2, \dots, m_f. \quad (\text{e 10.51})$$

Let  $P_n : B_n \rightarrow E_n$  be the projection map, and let  $\mathcal{G}'' \subset E_n$  be a finite subset such that  $\mathcal{G}'' \supset P(\mathcal{G}')$ .

Define  $\varphi'_{j,l} : M_{(n!)^2}(A(W, \alpha_n)) \rightarrow M_{(n!)^2}(R(\alpha_{n+1}, 1))$  be as defined (denoted by  $\varphi_{A,R,\alpha_{n+1}} \circ \Psi$  there) in 10.4 (with  $\mu_j = \lambda_{j,+}$  and  $\mu_i = 0$  if  $i \neq j$  (for  $\varphi'_{j,1}$ ); and with  $\mu_j = \lambda_{j,-}$  and  $\mu_i = 0$  if  $i \neq j$  (for  $\varphi'_{j,2}$ )) such that

$$[\varphi'_{j,l}(u_j)] = \exp(i2\pi H_{j,00}) \exp(i2\pi H_{j,0}) \exp(i2\pi H_{j,1}) \exp(i2\pi H_{j,2,l}), \quad (\text{e 10.52})$$

where  $H_{j,00}, H_{j,0}, H_{j,1}, H_{j,2,l} \in M_{(n!)^2}(R(\alpha_{n+1}, 1))$ ,  $l = 1, 2$ , such that

$$\text{tr}(H_{j,00}(t) + H_{j,0}(t) + H_{j,1}(t) + H_{j,2,l}(t)) = \text{tr}(H_{j,00}) + \text{tr}(H_{j,2,l}(t)), \quad l = 1, 2, \quad (\text{e 10.53})$$

$$\text{tr}(H_{j,2,1}(t)) = \lambda_+ h(t), \quad \text{tr}(H_{j,2,2}(t)) = \lambda_- h(t) \text{ and} \quad (\text{e 10.54})$$

$$|\text{tr}(H_{j,00}(t))| < \delta/4 \quad (\text{e 10.55})$$

for all  $t \in [0, 1]$ , where  $h(t)$  is  $C([0, 1])_+$  such that  $h(0) = 0$ ,  $h(3/4) = 1$ ,  $h(1) = 0$ ,  $h(t)$  is strictly increasing on  $[0, 3/4]$  and strictly decreasing on  $[3/4, 1]$ . Moreover  $\varphi'_{j,l}$  is  $\mathcal{G}''$ - $\delta/8(m_f)$ -multiplicative,

$$[\varphi'_{j,l}(u_i)] = \exp(i2\pi H_{i,00}) \exp(i2\pi H_{i,0}) \exp(i2\pi H_{i,1}), \text{ if } i \neq j \text{ and} \quad (\text{e 10.56})$$

$$[\varphi'_{j,l}]|_{\mathcal{Q}} = 0, \quad (\text{e 10.57})$$

where  $\mathcal{Q} = [P_n \circ \Phi_{n,\infty}](\mathcal{P}')$ . (Note that  $K_i(R(\alpha_{n+1}, 1)) = \{0\}, i = 0, 1$ ). Note since  $C \in \mathcal{D}$ , for each  $j$ , there exists a non-zero homomorphism  $\varphi''_{j,l} : M_{(n!)^2}(R(\alpha_{n+1}, 1)) \rightarrow C_{j,l} := \overline{c_{j,l}C}c_{j,l}$ ,  $j = 1, 2, \dots, m_f$ . It follows from 10.2 that there is, for each  $j$  and  $l$ , a homomorphism  $\varphi''_{j,l} : M_{(n!)^2}(R(\alpha_{n+1}, 1)) \rightarrow C_{j,l}$  such that

$$\sup\{|\tau \circ \varphi''_{j,1}(h) - g_{j,+}(\tau)| : \tau \in T(C)\} < \delta/2 \text{ and} \quad (\text{e 10.58})$$

$$\sup\{|\tau \circ \varphi''_{j,2}(h) - g_{j,-}(\tau)| : \tau \in T(C)\} < \delta/2. \quad (\text{e 10.59})$$

Let  $\varphi_{j,l} = \varphi''_{j,l} \circ \varphi'_{j,l} : E_n \rightarrow C_{j,l}$ . Recall  $\varphi'_{j,l}$  is of the form  $\varphi_{A,R,\alpha_{n+1}} \circ \Psi$ , we compute that (also using (e 10.45)),

$$D_{\tilde{C}}\left(\sum_{l=1}^2 [\varphi_{j,l}(u_j)]\right) \approx_{2\varepsilon_1/16(m_f)} (\lambda_{j,+}g_{j,+} - \lambda_{j,-}g_{j,-}) \quad (\text{in } \text{Aff}(T(\tilde{C}))/\mathbb{Z}) \quad (\text{e 10.60})$$

$$= \gamma(\bar{u}_j) \quad (\text{in } \text{Aff}(T(\tilde{C}))/\mathbb{Z}) \quad (\text{e 10.61})$$

$$D_{\tilde{C}}([\varphi_{j,l}(u_i)]) \approx_{2\varepsilon/16(m_f)} 0 \quad (\text{in } \text{Aff}(T(\tilde{C}))/\mathbb{Z}), \quad i \neq j. \quad (\text{e 10.62})$$

Similarly, using (e 10.46), we have

$$D_{\tilde{C}}\left(\sum_{l=1}^2(\lceil\varphi_{j,l}(u_{0,i})\rceil)\right) \approx_{2\varepsilon/16(m_f)} 0 \text{ (in } \text{Aff}(T(\tilde{C}))/\mathbb{Z}). \quad (\text{e } 10.63)$$

Now define  $\Phi' : E_n \rightarrow \bigoplus_{j=1}^{m_f} (\bigoplus_{l=1}^2 C_{j,l})$  by  $\Phi' = \sum_{j=1}^{m_f} (\sum_{l=1}^2 \varphi_{j,l})$ . From the above estimates,

$$\text{dist}(\Phi^\dagger(\bar{z}), \gamma(\bar{z})) < \eta_0 \text{ for all } z \in \mathcal{U}. \quad (\text{e } 10.64)$$

Moreover, since  $\Phi'$  factors through  $M_{(n!)^2}(R(\alpha_{n+1}, 1))$ ,

$$[\Phi']|_{\mathcal{Q}} = 0. \quad (\text{e } 10.65)$$

Define  $\Phi = \Phi' \circ P_n \circ L$ . We check that  $\Phi$  meets the requirements.  $\square$

**Lemma 10.6.** *Let  $C$  be a non-unital separable  $C^*$ -algebra. Suppose that  $u \in U(M_s(\tilde{C}))$  (for some integer  $s \geq 1$ ) with  $[u] \neq 0$  in  $K_1(C)$  but  $u^k \in CU(M_s(\tilde{C}))$  for some  $k \geq 1$ . Suppose that  $\pi_C(u) = e^{2i\pi\theta}$  for some  $\theta \in (M_s)_{s.a.}$ , where  $\pi_C : \tilde{C} \rightarrow \mathbb{C}$  is the quotient map. Then  $k\text{tr}(\theta) \in \mathbb{Z}$ , where  $\text{tr}$  is the tracial state of  $M_s$ .*

Let  $B$  be a stably projectionless simple separable  $C^*$ -algebra with  $\ker \rho_B = K_0(B)$  and with continuous scale. For any  $\varepsilon > 0$ , there exists  $\delta > 0$  and finite subset  $\mathcal{G} \subset C$  satisfying the following: If  $L_1, L_2 : C \rightarrow B$  are two  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps such that  $[L_1](u) = [L_2](u)$  in  $K_1(B)$ , then

$$\text{dist}(\overline{[L_1(u)]}, \overline{[L_2(u)]}) < \varepsilon. \quad (\text{e } 10.66)$$

*Proof.* Write  $u = e^{2i\pi\theta} + \zeta$ , where  $\zeta \in C_{s.a.}$  and  $\theta \in (M_s)_{s.a.}$ . Therefore, if  $u^k \in CU(\tilde{C})$ , then  $k\text{tr}(\theta) \in \mathbb{Z}$ .

Note  $L_i$  is originally defined on  $C$  and the extension  $L_i : \tilde{C} \rightarrow \tilde{B}$  has the property that  $L_i(u) = e^{2i\pi\theta} + L_i(\zeta)$ ,  $i = 1, 2$ .

Now write, for  $h_1, h_2, \dots, h_n \in \tilde{B}_{s.a.}$ ,

$$[L_1(u)] \cdot [L_2(u^*)] = \prod_{j=1}^n \exp(2i\pi h_j).$$

Then

$$\pi_B\left(\prod_{j=1}^n \exp(2i\pi h_j)\right) = 1.$$

By adding a self-adjoint scalar matrix to each  $h_j$ , we may assume, without loss of generality,  $h_j \in B_{s.a.}$  and that  $\sum_{j=1}^n \pi_B(h_j) = 0$ . It follows from 14.5 of [31] that, by choosing small  $\delta$  and large  $\mathcal{G}$  (independent of  $L_1$  and  $L_2$ ) there is  $h_0 \in \tilde{B}_{s.a.}$  such that  $\|h_0\| < \varepsilon/2(k+1)$  and

$$((\exp(2i\pi h_0))(\prod_{j=1}^n \exp(2i\pi h_j)))^k \in CU(\tilde{B}). \quad (\text{e } 10.67)$$

This also implies that  $\pi_B(h_0) = 0$ . Note also  $U(\tilde{B})/CU(\tilde{B}) = \text{Aff}(T(\tilde{B}))/\mathbb{Z}$ .

Note that, there is an integer  $m \in \mathbb{Z}$  such that, for any  $\tau \in T(\tilde{B})$ ,

$$k(\tau(\sum_{j=1}^n h_j + h_0)) = m. \quad (\text{e } 10.68)$$

Let  $t_0 \in T(\tilde{B})$  be defined by  $t_0(b) = \text{tr} \circ \pi_B(b)$  for all  $b \in \tilde{B}$ .

For any  $\tau_0 \in T(B)$  and any  $0 < \alpha < 1$ ,  $t = \alpha\tau_0 + (1 - \alpha)t_0$  is a tracial state of  $\tilde{B}$ . Then

$$kt\left(\sum_{j=1}^n h_j + h_0\right) = k(\alpha\tau_0\left(\sum_{j=1}^n h_j\right) + \alpha\tau_0(h_0)) = m. \quad (\text{e 10.69})$$

So  $k\alpha\tau_0(\sum_{j=1}^n h_j + h_0) = m$  for any  $0 < \alpha < 1$ . It follows that

$$\tau_0\left(\sum_{j=1}^n h_j + h_0\right) = 0 \text{ for all } \tau_0 \in T(\tilde{B}). \quad (\text{e 10.70})$$

□

## 11 Construction of homomorphisms

**Proposition 11.1.** *Let  $A$  be a separable simple  $C^*$ -algebra in  $\mathcal{D}$ . Suppose that  $\ker \rho_A = K_0(A)$ . Then there exists a sequence of approximately multiplicative completely positive contractive linear maps  $\{\varphi_n\}$  from  $A$  to  $W$  which maps strictly positive elements to strictly positive elements.*

*Proof.* Fix  $\tau \in T(A)$ . Define  $\gamma : T(W) \rightarrow T(A)$  by  $\gamma(t_W) = \tau$ , where  $t_W$  is the unique tracial state of  $W$ . Then  $\gamma$  induces an order semi-group homomorphism from  $\text{LAff}(\tilde{T}(A))$  onto  $\text{LAff}(\tilde{T}(W))$ . Since  $\ker \rho_A = K_0(A)$  and  $K_0(W) = 0$ , this in turn induces a homomorphism  $\Gamma : Cu^\sim(A) \rightarrow Cu^\sim(W)$ . Fix a strictly positive element  $a_0 \in A$  with  $\|a\| = 1$ . Let  $f_{a_0} > 0$  be the associated number (see 3.12). There exists a sequence of approximately multiplicative completely positive contractive linear maps  $\psi_n : A \rightarrow D_n$  such that  $\psi_n(a_0)$  is a strictly positive element of  $D_n$ ,  $t(f_{1/4}(a_0)) \geq f_{a_0}$  for all  $t \in T(D_n)$ . Moreover,

$$\lim_{n \rightarrow \infty} \sup\{|\tau(a) - \tau \circ \psi_n(a)| : \tau \in \overline{T(A)}^w\} = 0 \text{ for all } a \in A$$

(see the proof of 13.1 of [17]). In particular, this implies that  $\lim_{n \rightarrow \infty} \|\psi_n(x)\| = \|x\|$  for all  $x \in A$ . For each  $n$ , let  $\iota_n : D_n \rightarrow A$  be the embedding.

Let  $\lambda_n = \Gamma \circ (Cu^\sim(\iota_n))$ . It follows from [43] that there is a homomorphism  $h_n : D_n \rightarrow W$  such that

$$Cu^\sim(h_n) = \lambda_n, \quad n = 1, 2, \dots$$

By passing a subsequence if necessary, we may assume that

$$\lim_{n \rightarrow \infty} \|h_n \circ \psi_n(ab) - h_n \circ \psi_n(a)h_n \circ \psi_n(b)\| = 0 \text{ for all } a, b \in A.$$

By using an argument used in the proof 18.4 of [17], we can also assume that  $h_n \circ \psi_n(a_0)$  is a strictly positive element of  $W$ . □

**Remark 11.2.** Lemma 11.1 shows that every tracial state of simple  $C^*$ -algebras in  $\mathcal{D}$  with  $K_0(A) = \ker \rho_A$  is a  $W$ -trace. In the absence of the condition  $K_0(A) = \ker \rho_A$ , the proof of 11.1 shows that the conclusion of 11.1 holds if the assumption is changed to the assumption that  $A$  has at least one non-zero  $W$ -trace.

The following is a number theory lemma which may be known.

**Lemma 11.3.** *Let  $a_1, a_2, \dots, a_n$  be non-zero integers such that at least one of them is positive and one of them is negative. Then, for any  $d \in \mathbb{Z}$ , if  $a_1x_1 + a_2x_2 + \dots + a_nx_n = d$  has an integer solution, then it must have a positive integer solution.*

*Proof.* We will prove it by induction. Suppose that  $a, b \in \mathbb{Z}$  such that  $a > 0$  and  $b < 0$ . Suppose also there are  $x_0, y_0 \in \mathbb{Z}$  such that  $ax_0 + by_0 = d$ . Then, for any integer  $m \in \mathbb{Z}$ , and any  $x = x_0 + bm$  and  $y = y_0 - am$ ,

$$a(x_0 + bm) + b(y_0 - am) = d. \quad (\text{e 11.1})$$

Thus, by choosing negative integer  $m$  with large  $|m|$ , both  $x_0 + bm$  and  $y_0 - am$  are positive. This prove the case  $n = 2$ .

Suppose the lemma holds for  $n - 1$  for  $n \geq 3$ . Without lose of generality, let us first assume that  $a_1$  and  $a_2$  have different signs. Suppose  $x_1^0, x_2^0, \dots, x_n^0$  is an integer solution for  $a_1x_1 + a_2x_2 + \dots + a_nx_n = d$ , Let  $k = a_1x_1^0 + a_2x_2^0 + \dots + a_{n-1}x_{n-1}^0$ . Now we divided it into two cases:

Case 1:  $k$  and  $a_n$  have opposite signs. By induction assumption there are positive integers  $x'_1, x'_2, \dots, x'_{n-1}$  such that

$$k = a_1x'_1 + a_2x'_2 + \dots + a_{n-1}x'_{n-1}, \quad (\text{e 11.2})$$

since  $a_1a_2 < 0$  and  $n \geq 3$ . On the other hand, by applying the case  $n = 2$ , we have integers  $x > 0$  and  $y > 0$  such that  $kx + a_ny = d$ .

Let  $x_i = xx'_i$  for  $i \in \{1, 2, \dots, n-1\}$  and  $x_n = y$  to get desired positive integer solution for

$$\sum_{i=1}^n x_i a_i = d. \quad (\text{e 11.3})$$

Case 2:  $k$  and  $a_n$  have the same sign.

By the induction assumption there are positive integers:  $x'_1, x'_2, \dots, x'_{n-1}$  such that

$$-k = a_1x'_1 + a_2x'_2 + \dots + a_{n-1}x'_{n-1} \quad (\text{e 11.4})$$

(recall  $a_1a_2 < 0$ ). On the other hand apply the case  $n = 2$  (note that  $-k$  and  $a_n$  have opposite signs), we have  $x > 0$  and  $y > 0$  such that  $-kx + a_ny = d$ . Finally let  $x_i = xx'_i$  for  $i \in \{1, 2, \dots, n-1\}$  and  $x_n = y$  to get the desired positive integer solution.  $\square$

**11.4.** Recall from 6.2,  $\mathcal{Z}_0$  is an inductive limit of  $B_m = W_m \oplus M_{(m!)^2}(A(W, \alpha_m))$  and recall that  $K_0(\mathcal{Z}_0) = \mathbb{Z}$  and  $K_1(\mathcal{Z}_0) = \{0\}$ . Let  $E_m = M_{(m!)^2}(A(W, \alpha_m))$  be as in 6.2. For any  $m$ ,  $K_0(E_m) = \mathbb{Z}$  and  $K_1(E_m) = \{0\}$ . Let  $\text{id} : K_0(\mathcal{Z}_0) \cong K_0(E_m)$ . Then it induces a unique element in  $KK(\mathcal{Z}_0, E_m)$  and will be denote by  $\text{id}$ . Let  $z_{\mathbb{Z}} = [1] \in \mathbb{Z} = K_0(\mathcal{Z}_0)$  be the generator of  $K_0(\mathcal{Z}_0)$ . Suppose that  $C$  is a separable amenable  $C^*$ -algebra satisfies the UCT. Denote by  $(\kappa_{\mathcal{Z}_0})_{*i} : K_i(C \otimes \mathcal{Z}_0) \rightarrow K_i(C) \otimes \mathbb{Z} = K_i(C)$  the isomorphism such that  $(\kappa_{\mathcal{Z}_0})_{*i}(x \otimes z_{\mathbb{Z}}) = x$  for  $x \in K_i(C)$ , given by Kunneth's formula,  $i = 0, 1$ .

**Lemma 11.5.** *Let  $C \in \mathcal{D}$  be a separable simple  $C^*$ -algebra with continuous scale which satisfies the UCT. There exists a sequence of approximate multiplicative completely positive contractive linear maps  $\varphi_n : C \otimes \mathcal{Z}_0 \rightarrow C \otimes M_{k(n)}$  (for some subsequence  $\{k(n)\}$ ) which maps strictly positive elements to strictly positive elements such that*

$$[\varphi_n]|_{\mathcal{P}} = (\kappa_{\mathcal{Z}_0})|_{\mathcal{P}}, \quad (\text{e 11.5})$$

where  $\kappa_{\mathcal{Z}_0} \in KK(C \otimes \mathcal{Z}_0, C)$  is an invertible element which induces  $(\kappa_{\mathcal{Z}_0})_{*i}$ , for every finite subset  $\mathcal{P} \subset \underline{K}(C)$  and all sufficiently large  $n$ .

*Proof.* Let  $\varepsilon > 0$  and let  $\mathcal{F} \subset C$  be a finite subset.

Without loss of generality, we may assume that  $[L]|_{\mathcal{P}}$  is well-defined for any  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map from  $C$ . Without loss of generality, we may assume that  $\mathcal{P}$  generates the subgroup

$$G_{\mathcal{P}} \subset K_0(C) \bigoplus K_1(C) \bigoplus \bigoplus_{i=1,0} \bigoplus_{j=1}^m K_i(C, \mathbb{Z}/j\mathbb{Z}) \text{ for some } m \geq 2.$$

Let  $\delta > 0$  and  $\mathcal{G} \subset A$  be a finite subset. Let  $A$  be a unital simple AF-algebra with  $K_0(A) = \mathbb{Q} \oplus \mathbb{Z}$  and with  $\ker \rho_A = \mathbb{Z}$ . Write

$$A = \overline{\bigcup_{n=1}^{\infty} F_n},$$

where  $1_A \in F_n \subset F_{n+1}$  is a sequence of finite dimensional  $C^*$ -algebras. Recall that there is an identification of  $K_0(\mathcal{Z}_0)$  with  $\ker \rho_A \cong \mathbb{Z} \subset K_0(A)$ . Therefore there are sequences of pair of projections  $p_n, q_n \in F_n$  such that

$$(j_{n,\infty})_{*0}([p_n] - [q_n]) = z\mathbb{Z},$$

where  $j_{n,\infty} : F_n \rightarrow A$  is the embedding and  $z\mathbb{Z}$  is  $[1]$  in  $\mathbb{Z} \cong \ker \rho_A$ . Without loss of generality we may assume that

$$[p_n] \neq [q_n] \in K_0(F_n) \text{ for all } n \geq 1. \quad (\text{e 11.6})$$

Write

$$F_n = M_{k_1} \oplus M_{k_2} \oplus \cdots \oplus M_{k_l}.$$

Note that  $l \geq 3$  (see 7.7.2 of [2]). Let  $P_i : F_n \rightarrow M_{k_i}$  be the projection map. Let  $x_i = [P_i(p_n)] - [P_i(q_n)] \in \mathbb{Z}$ ,  $i = 1, 2, \dots, l$ . Then some of  $x_i > 0$  and some of  $x_i < 0$ . To see this, otherwise, we may assume that

$$x_i \geq 0 \text{ for all } i \in \{1, 2, \dots, l\}. \quad (\text{e 11.7})$$

Then  $[p_n] - [q_n] \geq 0$  for all  $n$ . It follows that, for all  $k \geq 1$ ,

$$(j_{n,n+k})_{*0}([p_n] - [q_n]) \geq 0 \text{ and } (j_{n,\infty})_{*0}([p_n] - [q_n]) \geq 0. \quad (\text{e 11.8})$$

That is  $(j_{n,\infty})_{*0}([p_n] - [q_n]) \in K_0(A)_+$ . This contradicts that  $(j_{n,\infty})_{*0}([p_n] - [q_n]) = z\mathbb{Z}$ .

Note that, as constructed in section 6, with  $A$  above,

$$\mathcal{Z}_0 = \lim_{m \rightarrow \infty} (E_m \oplus W_m), \quad (\text{e 11.9})$$

where  $W_m$  is a single summand of the form  $R(\alpha_m, 1)$  for some  $0 < \alpha_m < 1$  and  $E_m = M_{(m!)^2}(A(W, \alpha_m))$ . Note that  $K_i(W_m) = \{0\}$ ,  $i = 0, 1$ , and  $K_0(A(W, \alpha_m)) = \mathbb{Z}$  and  $K_1(A(W, \alpha_m)) = \{0\}$ . Let  $\text{id} \in KK(\mathcal{Z}_0, E_m)$  be as described in 11.4. Let  $\kappa_{00} \in KK(C \otimes \mathcal{Z}_0, C \otimes E_m)$  be the invertible element given by  $[\text{id}_C]$  and  $\text{id}$ .

By (e 11.9), there exists a  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $\Phi : C \otimes \mathcal{Z}_0 \rightarrow C \otimes E_m$  (for sufficiently large  $m$ ) such that

$$[\Phi]|_{\mathcal{P}} = (\kappa_{00})|_{\mathcal{P}} \quad (\text{e 11.10})$$

which maps strictly positive elements to strictly positive elements. Consider the short exact sequence

$$0 \rightarrow C_0((0, 1), Q) \rightarrow E_m \rightarrow M_{(m!)^2}(A) \rightarrow 0.$$



Let  $\varphi_{qa} : E_m \rightarrow M_{(m!)^2}(A)$  be the quotient map. Note that  $(\varphi_{qa})_{*0}$  gives an isomorphism from  $\mathbb{Z} = K_0(A(W, \alpha_m))$  onto  $\ker \rho_A$ . Let  $\varphi_q : C \otimes E_m \rightarrow C \otimes M_{(m!)^2}(A)$  be defined by  $\text{id}_C \otimes \varphi_{qa}$ . Let  $\varphi_1 : C \otimes \mathcal{Z}_0 \rightarrow C \otimes M_{(m!)^2}(A)$  be defined by  $\varphi_1 = \varphi_q \circ \Phi$ . For any  $\delta_1 > 0$  and finite subset  $\mathcal{F}_A \subset M_{(m!)^2}(A)$ , there is a unital  $\mathcal{F}_A$ - $\delta_1$ -multiplicative completely positive contractive linear map  $\Phi_A : M_{(m!)^2}(A) \rightarrow F_n$  (for some finite dimensional  $C^*$ -algebra  $F_n$ , here for above mentioned  $F_n$ , we still denote  $M_{(m!)^2}(F_n)$  by  $F_n$ ) such that  $[\Phi_A]|_{\ker \rho_A}$  is injective. Note that  $\Phi_A$  maps strictly positive elements of  $A$  to strictly positive elements of  $F_n$ . Write

$$F_n = M_{k_1} \oplus M_{k_2} \oplus \cdots \oplus M_{k_l}.$$

Let  $P_i : F_n \rightarrow M_{k_i}$  be the projection map. Let  $x_i = [P_i(p_n)] - [P_i(q_n)] \in \mathbb{Z}$ ,  $i = 1, 2, \dots, l$ . Without loss of generality, we may assume that

$$x_i > 0, \quad i = 1, 2, \dots, m^+ \quad \text{and} \quad x_i < 0, \quad i = m^+ + 1, \dots, l', \quad x_i = 0, \quad i = l' + 1, \dots, l. \quad (\text{e 11.11})$$

We claim that  $x_1, x_2, \dots, x_{l'}$  are relatively prime. If not,  $x_i = Nx'_i$ ,  $i = 1, 2, \dots, l'$ , for some  $N \geq 2$ . Then  $N(j_{n,\infty})_{*0}((x'_1, x'_2, \dots, x'_{l'})) = x_{\mathbb{Z}}$ . This is impossible since  $K_0(A) = \mathbb{Q} \oplus \mathbb{Z}$ . It follows from 11.3 that there are positive integers  $N_1, N_2, \dots, N_l$  such that

$$\sum_{i=1}^l N_i x_i = 1. \quad (\text{e 11.12})$$

Let  $r = \sum_{i=1}^l N_i k_i$ . Define  $\iota : F_k \rightarrow M_r$  by

$$\iota((f_1, f_2, \dots, f_l)) = \bigoplus_{i=1}^l \iota_i(f_i), \quad (\text{e 11.13})$$

where  $\iota_i : M_{k_i} \rightarrow M_r$  is defined by

$$\iota_i(f_i) = \text{diag}(\overbrace{f_i, f_i, \dots, f_i}^{N_i}) \quad \text{for all } f_i \in M_{k_i}, \quad i = 1, 2, \dots, l. \quad (\text{e 11.14})$$

Let  $\kappa_{\mathcal{Z}_0} \in KL(C \otimes \mathcal{Z}_0, C)$  be defined by, for  $j = 2, 3, \dots$ ,

$$\kappa_{\mathcal{Z}_0}(x \otimes z_{\mathbb{Z}}) = x \quad \text{for all } x \in K_i(C \otimes \mathcal{Z}_0) \oplus K_i(C \otimes \mathcal{Z}_0, \mathbb{Z}/j\mathbb{Z}), \quad i = 0, 1. \quad (\text{e 11.15})$$

Note that  $(\iota_{*0})([p_n] - [q_n]) = [1] \in \mathbb{Z} = K_0(M_r)$ . Let  $L = (\text{id}_C \otimes \iota) \circ (\text{id}_C \otimes \Phi_A) \circ \varphi_1 : C \otimes \mathcal{Z}_0 \rightarrow C \otimes M_r$ . By choosing  $\delta$  and  $\delta_1$  sufficiently small and  $\mathcal{G}$  and  $\mathcal{G}_A$  sufficiently large,  $L$  is  $\mathcal{F}$ - $\varepsilon$ -multiplicative. Moreover, we compute that

$$[L]|_{\mathcal{P}} = [\kappa_{\mathcal{Z}_0}]|_{\mathcal{P}}.$$

□

**Lemma 11.6.** *Let  $A$  and  $B$  be non-unital separable simple  $C^*$ -algebras in  $\mathcal{D}$  with  $K_0(A) = \ker \rho_A$  and with  $K_0(B) = \ker \rho_B$ , respectively, which have continuous scale and satisfy the UCT. Suppose that there is  $\kappa \in KL(A, B)$  and an affine continuous map  $\kappa_T : T(B) \rightarrow T(A)$ . Then, there exists a sequence of approximate multiplicative completely positive contractive linear maps  $\varphi_n : A \rightarrow B$  such that*

$$[\{\varphi_n\}] = \kappa \quad \text{and} \quad (\text{e 11.16})$$

$$\lim_{n \rightarrow \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \quad \text{for all } a \in A_{s.a.}. \quad (\text{e 11.17})$$

*Proof.* Let  $\varepsilon > 0, \eta > 0$ ,  $\mathcal{F} \subset A$  be a finite subset and  $\mathcal{H} \subset A_{s,a}$  be a finite subset.

Fix a finite subset  $\mathcal{P} \subset \underline{K}(A)$ . We may assume that, for some  $m \geq 1$ ,

$$\mathcal{P} \subset K_0(A) \bigoplus K_1(A) \bigoplus_{j=1}^m (K_0(A, \mathbb{Z}/j\mathbb{Z}) \oplus K_1(A, \mathbb{Z}/j\mathbb{Z})).$$

Moreover,  $m!x = 0$  for all  $x \in \text{Tor}(K_0(A)) \cap \mathcal{P}$ . Let  $G_{0,\mathcal{P}}$  be the subgroup generated by  $K_0(A) \cap \mathcal{P}$ . We may write  $G_{0,\mathcal{P}} := F_0 \oplus G_0$ , where  $F_0$  is free and  $G_0$  is torsion. In particular,  $m!x = 0$  for all  $x \in G_0$ .

Choose  $\delta > 0$  and finite subset  $\mathcal{G} \subset A$  so that  $[L]|_{\mathcal{P}}$  is well defined for any  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $L$  from  $A$ . We may assume that  $\delta < \varepsilon$  and  $\mathcal{F} \cup \mathcal{H} \subset \mathcal{G}$ . Since both  $A$  and  $B$  have continuous scale,  $T(A)$  and  $T(B)$  are compact (9.3 of [17]).

Choose  $a_0 \in A_+$  such that  $\|a_0\| = 1$  and

$$d_\tau(a_0) < \min\{\eta, \delta\}/4 \text{ for all } \tau \in T(A). \quad (\text{e 11.18})$$

Let  $e_0 \in A$  be a strictly positive element of  $A$  with  $\|e_0\| = 1$  such that  $\tau(e_0) > 15/16$  for all  $\tau \in T(A)$ .

Since  $A \in \mathcal{D}_0$  (see 7.8), by 14.8 of [17], there are  $\mathcal{G}$ - $\delta/4$ -multiplicative completely positive contractive linear maps  $\varphi_0 : A \rightarrow \overline{\varphi_0(A)A\varphi_0(A)}$  and  $\psi_0 : A \rightarrow D \subset A$  with  $D \in \mathcal{C}_0^{0'}$  such that

$$\|x - \text{diag}(\varphi_0(x), \overbrace{\psi_0(x), \psi_0(x), \dots, \psi_0(x)}^{m!})\| < \delta/16 \text{ for all } x \in \mathcal{G}, \quad (\text{e 11.19})$$

$$\varphi_0(e_0) \lesssim a_0, \quad (\text{e 11.20})$$

$$t(f_{1/4}(\psi_0(e_0))) > 1/4 \text{ for all } t \in T(D). \quad (\text{e 11.21})$$

Let  $\Psi_0 : A \rightarrow M_{m!}(D) \subset A$  be defined by

$$\Psi_0(a) = \text{diag}(\overbrace{\psi_0(x), \psi_0(x), \dots, \psi_0(x)}^{m!}) \text{ for all } a \in A. \quad (\text{e 11.22})$$

Let  $\mathcal{P}_1 = [\varphi_0](\mathcal{P})$  and  $\mathcal{P}_2 = [\Psi_0](\mathcal{P})$ . Put  $\mathcal{P}_3 = \mathcal{P} \cup \mathcal{P}_1 \cup \mathcal{P}_2$ . Note that, since  $K_i(D) = \{0\}$  ( $i = 0, 1$ ),  $\Psi_0|_{\mathcal{P} \cap K_i(A)} = 0$ ,  $i = 0, 1$ . Moreover, by (e 11.22),

$$[\Psi_0]|_{\mathcal{P} \cap K_i(\mathbb{Z}/j\mathbb{Z})} = 0, \quad i = 0, 1, \quad j = 2, \dots, m. \quad (\text{e 11.23})$$

Set

$$d = \inf\{d_\tau(\varphi_0(e_0)) : \tau \in T(A)\}. \quad (\text{e 11.24})$$

We also have

$$[\varphi_0]|_{F_0} = [\text{id}_A]|_{F_0}. \quad (\text{e 11.25})$$

Let  $\mathcal{G}_1 = \mathcal{G} \cup \varphi_0(\mathcal{G})$ . Choose  $0 < \delta_1 < \delta$  and finite subset  $\mathcal{G}_1 \subset A$  such that  $[L']|_{\mathcal{P}_4}$  is well defined for any  $\mathcal{G}_1$ - $\delta_1$ -multiplicative completely positive contractive linear map from  $A$ .

It follows from 9.8, 11.1 and 11.5 that there exists a  $\mathcal{G}_1$ - $\delta_1/4$ -multiplicative completely positive contractive linear map  $L : A \rightarrow B \otimes M_K$  for some integer  $K$  such that

$$[L]|_{\mathcal{P}_3} = \kappa_{\mathbb{Z}_0} \circ \kappa_{\mathbb{Z}_0}^{-1} \circ \kappa|_{\mathcal{P}_3} = \kappa|_{\mathcal{P}_3}. \quad (\text{e 11.26})$$

Without loss of generality, we may assume that  $\mathcal{G}_1 \subset A^1$ .

Let  $b_0 \in B$  with  $\|b_0\| = 1$  such that

$$\tau(b_0) < \min\{\eta, \delta_1, d\}/16(K+1) \text{ for all } \tau \in T(B). \quad (\text{e 11.27})$$

Let  $e_b \in B \otimes M_K$  be a strictly positive element of  $B \otimes M_K$  such that

$$\tau(e_b) > 7/8 \text{ for all } \tau \in T(B \otimes M_K). \quad (\text{e 11.28})$$

Let  $Q \subset \underline{K}(B)$  be a finite subset which contains  $[L](\mathcal{P}_4)$ . We assume that

$$Q \subset K_0(B) \bigoplus K_1(B) \bigoplus \bigoplus_{i=0,1} \bigoplus_{j=1}^{m_1} K_i(B, \mathbb{Z}/j\mathbb{Z}) \quad (\text{e 11.29})$$

for some  $m_1 \geq 2$ . Moreover, we may assume that  $m_1 x = 0$  for all  $x \in \text{Tor}(G_{0,b})$ , where  $G_{0,b}$  is the subgroup generated by  $Q \cap K_0(B)$ . Without loss of generality, we may assume that  $m|m_1$ .

Let  $\mathcal{G}_b \subset B \otimes M_K$  be a finite subset and  $1/2 > \delta_2 > 0$  be such that  $[\Phi]|_Q$  is well defined for any  $\mathcal{G}_b$ - $\delta_2$ -multiplicative completely positive contractive linear map  $\Phi$  from  $B \otimes M_K$ . Note also, by 7.8,  $B \in \mathcal{D}_0$ . There are  $\mathcal{G}_b$ - $\delta_2$ -multiplicative completely positive contractive linear maps  $\varphi_{0,b} : B \otimes M_K \rightarrow \varphi_{0,b}(B \otimes M_K)(B \otimes M_K)\varphi_{0,b}(B \otimes M_K)$  and  $\psi_{0,b} : B \otimes M_K \rightarrow D_b \subset B \otimes M_K$  with  $D_b \in \mathcal{C}_0^{0'}$  such that

$$\|b - \text{diag}(\varphi_{0,b}(b), \overbrace{\psi_{0,b}(b), \psi_{0,b}(b), \dots, \psi_{0,b}(b)}^{(m_1)!})\| < \min\{\delta_2, \varepsilon/16, \eta/16\} \text{ for all } b \in \mathcal{G}_b \quad (\text{e 11.30})$$

$$\text{and } \varphi_{0,b}(e_b) \lesssim b_0 \text{ and } t(\psi_{0,b}) > 3/4 \text{ for all } t \in T(D_b). \quad (\text{e 11.31})$$

Note that  $K_1(D_b) = \{0\} = K_0(D_b)$ . Moreover, as in (e 11.19) and (e 11.23), we may also assume that

$$[\psi_{0,b}]|_{\text{Tor}(G_{0,b})} = 0 \text{ and } [\psi_{0,b}]|_{Q \cap K_i(B, \mathbb{Z}/j\mathbb{Z})} = 0, \quad j = 2, 3, \dots, m_1. \quad (\text{e 11.32})$$

Therefore

$$[\varphi_{0,b}]|_{\text{Tor}(G_{0,b})} = [\text{id}_B]|_{\text{Tor}(G_{0,b})}, [\varphi_{0,b}]|_{Q \cap K_1(B)} = [\text{id}_B]|_{Q \cap K_1(B)} \text{ and} \quad (\text{e 11.33})$$

$$[\varphi_{0,b}]|_{Q \cap K_i(B, \mathbb{Z}/j\mathbb{Z})} = [\text{id}_B]|_{Q \cap K_i(B, \mathbb{Z}/j\mathbb{Z})}, \quad j = 2, 3, \dots, m_1. \quad (\text{e 11.34})$$

Let  $G_{\mathcal{P}}$  be the subgroup generated by  $\mathcal{P}$  and let  $\kappa' = \kappa - [\varphi_{0,b}] \circ [L] \circ [\varphi_0]$  be defined on  $G_{\mathcal{P}}$ .

Then, by (e 11.26), (e 11.33) and (e 11.34), we compute that

$$\kappa'|_{G_{0,\mathcal{P}}} = 0, \quad \kappa'|_{\mathcal{P} \cap K_1(A)} = 0 \text{ and} \quad (\text{e 11.35})$$

$$\kappa'|_{\mathcal{P} \cap K_i(A, \mathbb{Z}/j\mathbb{Z})} = 0, \quad j = 2, 3, \dots, m. \quad (\text{e 11.36})$$

Let  $\iota : M_{m!}(D) \rightarrow A$  be the embedding.

Let  $\kappa_T^\sharp : \text{Aff}(T(A)) \rightarrow \text{Aff}(T(B))$ . This induces an order semigroup homomorphism  $\tilde{\kappa}^T : \text{LAff}_+(\tilde{T}(A)) \rightarrow \text{LAff}_+(\tilde{T}(B))$ . By 7.6, one checks easily that  $\kappa_T^\sharp$  is a Cuntz semigroup homomorphism.

Let  $\gamma' : Cu(M_{m!}(D)) \rightarrow \text{LAff}_+(\tilde{T}(B))$  be the Cuntz semi-group homomorphism given by  $\gamma' = \kappa_T^\sharp \circ Cu(\iota)$ . Put  $\gamma : Cu(M_{m!}(D)) \rightarrow \text{LAff}_+(\tilde{T}(B))$  defined by  $\gamma(f) = (1 - \min\{\eta, \eta_0\}/2(m!))\gamma'(f)$  for all  $f \in Cu(M_{m!}(D))$ .

Let  $\gamma_0 : Cu^\sim(M_{m!}(D)) \rightarrow Cu^\sim(B)$  be the morphism induced by  $\gamma$  (note  $K_0(M_{m!}(D)) = \{0\}$ ).

By applying 1.0.1 of [43], one obtains a homomorphism  $h_d : M_{m!}(D) \rightarrow B$  such that

$$(h_d)_*0 = \gamma_{00} \text{ and } \tau \circ h_d(c) = \gamma(\widehat{c})(\tau) \text{ for all } \tau \in T(B) \text{ and } c \in (M_{m!}(D))_{s.a.} \quad (\text{e 11.37})$$

Define  $h : A \rightarrow B$  by  $h = h_d \circ \Psi_0$ . Then

$$[h]|_{\mathcal{P}} = \kappa'|_{\mathcal{P}}, [h]|_{\mathcal{P} \cap K_1(A)} = 0 \text{ and } [h]|_{\mathcal{P} \cap K_i(\mathbb{Z}/j\mathbb{Z})} = 0, \quad i = 2, 3, \dots, m. \quad (\text{e 11.38})$$

Moreover,

$$\tau(h(a)) = \gamma(\widehat{\Psi_0(a)}) \text{ for all } a \in A \text{ and } \tau \in T(B). \quad (\text{e 11.39})$$

Let  $e_d \in M_m(D)$  be a strictly positive element with  $\|e_d\| = 1$ . Then, by (e 11.24),

$$d_\tau(h_d(e_d)) < 1 - d \text{ for all } \tau \in T(B). \quad (\text{e 11.40})$$

It follows from (e 11.27) that

$$d_\tau(h(e_d)) + d_\tau(\varphi_{0,b}(e_0)) < 1 \text{ for all } \tau \in T(B). \quad (\text{e 11.41})$$

Note that  $B$  has stable rank one (see 15.5 of [17]). By omitting conjugating a unitary in  $B$  without loss of generality, we may assume that  $\varphi_{0,b} \circ L \oplus h$  maps  $A$  into  $B$ . Put  $\Phi = \varphi_{0,b} \circ L \oplus h$ . Then  $\Phi$  is  $\mathcal{G}$ - $\delta$ -multiplicative. Moreover, we compute that

$$[\Phi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and } \sup\{|\tau(\Phi(x)) - \kappa_T(\tau)(x)| : \tau \in T(B_T)\} < \eta \text{ for all } x \in \mathcal{H}. \quad (\text{e 11.42})$$

The lemma then follows.  $\square$

**Lemma 11.7.** *Let  $A$  be a non-unital simple separable  $C^*$ -algebra in  $\mathcal{D}$  with  $K_0(A) = \ker \rho_A$  and with continuous scale which satisfies the UCT. Let  $B_T$  be as in 6.2. Suppose that there is  $\kappa \in KL(B_T, A)$ , an affine continuous map  $\kappa_T : T(A) \rightarrow T(B_T)$  and a continuous homomorphism  $\kappa_{uc} : U(\tilde{B}_T)/CU(\tilde{B}_T) \rightarrow U(\tilde{A})/CU(\tilde{A})$  such that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible. Then there exists a sequence of approximate multiplicative completely positive contractive linear maps  $\varphi_n : B_T \rightarrow A$  such that*

$$[\{\varphi_n\}] = \kappa \quad (\text{e 11.43})$$

$$\lim_{n \rightarrow \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \text{ for all } a \in A_{s.a.} \text{ and } \quad (\text{e 11.44})$$

$$\lim_{n \rightarrow \infty} \text{dist}(\kappa_{uc}(z), \varphi_n^\dagger(z)) = 0 \text{ for all } z \in U(\tilde{A})/CU(\tilde{A}). \quad (\text{e 11.45})$$

*Proof.* Let  $\varepsilon > 0$ , let  $\eta > 0$  and let  $\sigma > 0$ , let  $\mathcal{P} \subset \underline{K}(B_T)$  be a finite subset, let  $S_u \subset U(\tilde{B}_T)/CU(\tilde{B}_T)$  be a finite subset, let  $\mathcal{H} \subset (B_T)_{s.a.}$  be a finite subset and let  $\mathcal{F} \subset B_T$  be a finite subset.

Without loss of generality, we may assume that  $\mathcal{F} \subset (B_T)^1$  and  $[L']|_{\mathcal{P}}$  and  $(L')^\dagger|_{S_u}$  are well-defined for any  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map from  $B_T$ .

Let  $G_1 \subset K_1(B_T)$  be the subgroup generated by  $\mathcal{P} \cap K_1(B_T)$ .

Fix  $\delta > 0$  and a finite subset  $\mathcal{G} \subset B_T$ . We assume that  $\delta < \min\{\varepsilon/2, \eta/4, \sigma/16\}$ . To simplify notation, without loss of generality, we may assume that  $G_1 \subset F \subset (\Phi_{n_0, \infty})_{*1}(K_1(B_{n_0}))$  for some  $n_0 \geq 1$ , where  $F$  is a finitely generated standard subgroup (see 10.3). We also choose  $n_0$  larger than that required by 10.5 for  $\delta$  (in place of  $\varepsilon$ )  $\mathcal{G}$  (in place of  $\mathcal{F}$ )  $\mathcal{P}$  and  $\sigma/16$  (in place of  $\delta_0$ ).

Without loss of generality, we may write

$$S_u = S_{u,1} \sqcup S_{u,0}, \quad (\text{e 11.46})$$

where  $S_{u,1} \subset J_{F,u}(F)$  and  $S_{u,0} \subset U_0(\tilde{B}_T)/CU(\tilde{B}_T) = \text{Aff}(T(B_T))/\mathbb{Z}$  and both  $S_{u,1}$  and  $S_{u,0}$  are finite subsets. For  $w \in S_{u,0}$ , write

$$w = \prod_{j=1}^{l(w)} \exp(i2\pi h_{w,j}), \quad (\text{e 11.47})$$

where  $h_{w,j} \in (B_T)_{s.a.}$ ,  $j = 1, 2, \dots, l(w)$ . Let

$$\mathcal{H}_u = \{h_{w,j} : 1 \leq j \leq l(w), w \in S_{u,0}\} \text{ and } M = \max\left\{\sum_{i=1}^{l(w)} \|h_{w,i}\| : w \in S_{u,0}\right\}. \quad (\text{e 11.48})$$

To simplify notation further, we may assume that  $G_1 = F$ .

Write  $G_1 = \mathbb{Z}^{m_f} \oplus \text{Tor}(G_1)$  and  $\mathbb{Z}^{m_f}$  is generated by cyclic and free generators  $x_1, x_2, \dots, x_{m_f}$ . Let  $\text{Tor}(G)$  be generated by  $x_{0,1}, x_{0,2}, \dots, x_{0,m_t}$ . Let  $u_1, u_2, \dots, u_{m_f}, u_{1,0}, u_{2,0}, \dots, u_{m_t,0} \in U(\tilde{B}_T)$  be unitaries such that  $[u_i] = x_i$ ,  $i = 1, 2, \dots, m_f$ , and  $[u_{j,0}] = x_{0,j}$ ,  $j = 1, 2, \dots, m_t$ . Let  $\pi_u : U(\tilde{B}_T)/CU(\tilde{B}_T) \rightarrow K_1(B_T)$  be the quotient map and let  $G_u$  be the subgroup generated by  $S_{u,1}$ . Since  $(\kappa, \kappa_T, \kappa_u)$  is compatible, without loss of generality, we may assume that  $\pi_u(G_u) = \{x_1, x_2, \dots, x_{m_f}\} \cup \{x_{0,1}, x_{0,2}, \dots, x_{0,m_t}\}$  and  $S_{u,1} = \{\bar{u}_1, \bar{u}_2, \dots, \bar{u}_{m_f}, \bar{u}_{1,0}, \bar{u}_{2,0}, \dots, \bar{u}_{m_t,0}\}$  as described in 10.3, in particular,  $k_j \bar{u}_{j,0} = 0$  in  $U(\tilde{B}_T)/CU(\tilde{B}_T)$ ,  $j = 1, 2, \dots, m_t$ .

Let  $\varphi_n : B_T \rightarrow A$  be a sequence of approximately multiplicative completely positive contractive linear maps given by 11.6 such that

$$[\{\varphi_n\}] = \kappa \text{ and} \quad (\text{e 11.49})$$

$$\lim_{n \rightarrow \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \text{ for all } a \in A_{s.a.}. \quad (\text{e 11.50})$$

Fix a strictly positive element  $e_b \in B_T$  with  $\|e_b\| = 1$  and  $\tau(e_b) \geq 15/16$  and  $\tau(f_{1/2}(e_b)) \geq 15/16$  for all  $\tau \in T(B_T)$ .

Let  $\mathcal{F}_b \subset B_T$  be a finite subset which contains  $\mathcal{F} \cup \mathcal{H} \cup \mathcal{H}_u$ . and let  $\delta_b > 0$ . There are  $\mathcal{F}_b$ - $\delta_b$ -multiplicative completely positive contractive linear maps  $\Phi_0 : B_T \rightarrow D_b \subset B_T$  with  $D_b \in C_0^0$  and  $\Phi_1 : B_T \rightarrow B_T$  such that

$$\|b - \text{diag}(\Phi_0(b), \Phi_1(b))\| < \delta_b/2 \text{ for all } b \in \mathcal{F}_b \text{ and} \quad (\text{e 11.51})$$

$$0 < d_\tau(\Phi_0(e_b)) < \min\{\eta, \sigma/16\}/4(M+1) \text{ for all } \tau \in T(B_T). \quad (\text{e 11.52})$$

(see, for example 14.8 of [17]). Note that  $K_0(D_b) = K_1(D_b) = \{0\}$ . Therefore, for any sufficiently large  $n$ ,

$$[\varphi_n \circ \Phi_0]|_{\mathcal{P}} = 0, \quad [\varphi_n \circ \Phi_1]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and} \quad (\text{e 11.53})$$

$$d_\tau(\varphi_n(\Phi_0(e_b))) < \min\{\eta, \sigma/16\}/2(M+1) \text{ for all } \tau \in T(A). \quad (\text{e 11.54})$$

Fix a sufficiently large  $n$ . Define  $\lambda = \kappa|_{G_u} - (\varphi_n \circ \Phi_1)^\dagger|_{G_u} : G_u \rightarrow U(\tilde{A})/CU(\tilde{A})$ . Since  $(\kappa, \kappa_T, \kappa_u)$  is compatible,  $\pi_u \circ \lambda(\bar{u}_i) = 0$  and  $\pi_u \circ \lambda(\bar{u}_{0,j}) = 0$ ,  $i = 1, 2, \dots, m_f$  and  $j = 1, 2, \dots, m_t$ .

Let  $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{H}$ . It follows from 10.5 that there exists  $\mathcal{F}_1$ - $\min\{\varepsilon/4, \eta/4\}$ -multiplicative completely positive contractive linear map  $L : B_T \rightarrow \overline{cAc}$ , where  $c = \varphi_n \circ \Phi_0(e_b)$ , such that

$$[L]|_{\mathcal{P}} = 0 \text{ and } \text{dist}(L^\dagger(\bar{u}_j), \lambda(\bar{u}_j)) < \sigma, \quad j = 1, 2, \dots, m_f. \quad (\text{e 11.55})$$

Define  $\Psi : A \rightarrow B_T$  by

$$\Psi(a) = \text{diag}(L(a), \Phi_1 \circ \varphi_n(a)) \text{ for all } a \in A. \quad (\text{e 11.56})$$

Then  $\Psi$  is  $\mathcal{F}$ - $\varepsilon$ -multiplicative if  $n$  is sufficiently large. It follows from (e 11.53), (e 11.55) and the definition of  $\lambda$  that

$$[\Psi]|_{\mathcal{P}} = \kappa|_{\mathcal{P}} \text{ and } \text{dist}(\Psi^\dagger(\bar{u}_j), \kappa_{uc}(\bar{u}_j)) < \sigma, \quad j = 1, 2, \dots, m_f. \quad (\text{e 11.57})$$

By 10.6, we may also have

$$\text{dist}(\Psi^\dagger(\bar{u}_{j,0}), \kappa_{uc}(\bar{u}_{j,0})) < \sigma, \quad j = 1, 2, \dots, m_t. \quad (\text{e 11.58})$$

By the choice of  $M$  and  $\mathcal{H}_u$ , (e 11.52) and by the assumption that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible,

$$\text{dist}(\Psi^\dagger(\bar{w}), \kappa_{u,c}(\bar{w})) < \sigma \text{ for all } w \in S_{u,0}. \quad (\text{e 11.59})$$

Moreover, by (e 11.52), by (e 11.50) and by choosing sufficiently large  $n$ ,

$$\sup\{|\tau(\Psi(b)) - \kappa_T(\tau)(b)| : \tau \in T(A)\} < \eta \text{ for all } b \in \mathcal{H}. \quad (\text{e 11.60})$$

□

**Theorem 11.8.** *Let  $A$  be a separable amenable simple  $C^*$ -algebra in  $\mathcal{D}_0$  with continuous scale which satisfies the UCT. Let  $B_T$  be as in 6.2. Suppose that there is  $\kappa \in KL(B_T, A)$ , an affine continuous map  $\kappa_T : T(A) \rightarrow T(B_T)$  and a continuous homomorphism  $\kappa_{uc} : U(\tilde{B}_T)/CU(\tilde{B}_T) \rightarrow U(\tilde{A})/CU(\tilde{A})$  such that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible. Then there exists a homomorphism  $\varphi : B_T \rightarrow A$  such that*

$$[\varphi] = \kappa, \quad \tau \circ \varphi(a) = \kappa_T(\tau)(a) \text{ for all } a \in A_{s.a.} \text{ and } \varphi^\dagger = \kappa_{uc}. \quad (\text{e 11.61})$$

*Proof.* Let  $e_b \in B_T$  be a strictly positive element of  $B_T$  with  $\|e_b\| = 1$ . Since  $A$  has continuous scale, without loss of generality, we may assume that

$$\min\{\inf\{\tau(e_b) : \tau \in T(B_T)\}, \inf\{\tau(f_{1/2}(e_b)) : \tau \in T(B_T)\}\} > 3/4. \quad (\text{e 11.62})$$

Let  $T : (B_T)_+ \setminus \{0\} \rightarrow \mathbb{N} \times \mathbb{R}_+ \setminus \{0\}$  be given by Theorem 9.6 of [17].

By 11.7, there exists a sequence of approximately multiplicative completely positive contractive linear maps  $\varphi_n : B_T \rightarrow A$  such that

$$[\{\varphi_n\}] = \kappa \quad (\text{e 11.63})$$

$$\lim_{n \rightarrow \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)| : \tau \in T(A)\} = 0 \text{ for all } a \in (B_T)_{s.a.} \text{ and } \quad (\text{e 11.64})$$

$$\lim_{n \rightarrow \infty} \text{dist}(\kappa_{uc}(z), \varphi_n^\dagger(z)) = 0 \text{ for all } z \in U(\tilde{B}_T)/CU(\tilde{B}_T). \quad (\text{e 11.65})$$

Let  $\varepsilon > 0$  and  $\mathcal{F} \subset B_T$  be a finite subset.

We will apply 5.3. Note that  $K_0(\tilde{A})$  is weakly unperforated (see 5.5 and 7.10).  $\delta_{1,1} > 0$  (in place of  $\delta$ ),  $\gamma_1 > 0$  (in place of  $\gamma$ ),  $\eta_1 > 0$  (in place of  $\eta$ ), let  $\mathcal{G}_{1,1} \subset B_T$  (in place of  $\mathcal{G}$ ) be a finite subset,  $\mathcal{H}_{1,1} \subset (B_T)_+ \setminus \{0\}$  (in place of  $\mathcal{H}_1$ ) be a finite subset,  $\mathcal{P}_1 \subset \underline{K}(B_T)$  (in place of  $\mathcal{P}$ ),  $\mathcal{U}_1 \subset U(\tilde{U})$  (in place of  $\mathcal{U}$ ) with  $\bar{\mathcal{U}} = \mathcal{P} \cap K_1(B_T)$  and let  $\mathcal{H}_{1,2} \subset (B_T)_{s.a.}$  (in place of  $\mathcal{H}_2$ ) required by Theorem 5.3 for  $T$ ,  $\varepsilon$  and  $\mathcal{F}$  (with  $T(k, n) = n$ , see 5.2).

Without loss of generality, we may assume that  $\mathcal{H}_{1,1} \subset (B_T)_+^1 \setminus \{0\}$  and  $\gamma_1 < 1/64$ .

Let  $\mathcal{G}_{1,2} \subset B_T$  (in place of  $\mathcal{G}$ ) be a finite subset and let  $\delta_{1,2} > 0$  be required by Theorem 9.6 of [17] for the above  $\mathcal{H}_{1,1}$  (in place of  $\mathcal{H}_1$ ). Let  $\delta_1 = \min\{\delta_{1,1}, \delta_{1,2}\}$  and  $\mathcal{G}_1 = \mathcal{G}_{1,1} \cup \mathcal{G}_{1,2}$ .

Choose  $n_0 \geq 1$  such that  $\varphi_n$  is  $\mathcal{G}_1$ - $\delta_1/2$ -multiplicative, for all  $n \geq n_0$ ,

$$[\varphi_n]|_{\mathcal{P}_1} = \kappa|_{\mathcal{P}_1}, \quad (\text{e 11.66})$$

$$\sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)| : \tau \in T(B_T)\} < \gamma_1/2 \text{ for all } a \in \mathcal{H}_{1,2}, \quad (\text{e 11.67})$$

$$\tau(f_{1/2}(\varphi_n(e_a))) > 3/8 \text{ for all } \tau \in T(B_T) \text{ and } \quad (\text{e 11.68})$$

$$\text{dist}(\varphi_n^\dagger(\bar{u}), \kappa_{uc}(\bar{u})) < \eta/2 \text{ for all } u \in \mathcal{U}. \quad (\text{e 11.69})$$

By applying 9.6 of [17],  $\varphi_n$  are all  $T$ - $\mathcal{H}_{1,1}$ -full. By applying Theorem 5.3, we obtain a unitary  $u_n \in \tilde{B}_T$  (for each  $n \geq n_0$ ) such that

$$\|u_n^* \varphi_n(a) u_n - \varphi_{n_0}(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad (\text{e 11.70})$$

Now let  $\{\varepsilon_n\}$  be an decreasing sequence of positive elements such that  $\sum_{n=1}^{\infty} \varepsilon_n < \infty$  and let  $\{\mathcal{F}_n\}$  be an increasing sequence of finite subsets of  $B_T$  such that  $\cup_{n=1}^{\infty} \mathcal{F}_n$  is dense in  $B_T$ .

By what have been proved, we obtain a subsequence  $\{n_k\}$  and a sequence of unitaries  $\{u_k\} \subset \tilde{B}_T$  such that

$$\|\text{Ad } u_{k+1} \circ \varphi_{n_{k+1}}(a) - \text{Ad } u_k \circ \varphi_{n_k}(a)\| < \varepsilon_k \text{ for all } a \in \mathcal{F}_k, \quad (\text{e 11.71})$$

$k = 1, 2, \dots$ . Since  $\cup_{n=1}^{\infty} \mathcal{F}_n$  is dense in  $B_T$ , by (e 11.71),  $\{\text{Ad } u_k \circ \varphi_{n_k}(a)\}$  is a Cauchy sequence. Let

$$\varphi(a) = \lim_{k \rightarrow \infty} \text{Ad } u_k \circ \varphi_{n_k}(a) \text{ for all } a \in B_T. \quad (\text{e 11.72})$$

Then  $\varphi : B_T \rightarrow A$  is a homomorphism which satisfies (e 11.61).  $\square$

**Lemma 11.9.** *Let  $A$  be a non-unital simple separable  $C^*$ -algebra in  $\mathcal{D}$  with  $K_0(A) = \ker \rho_A$  and with continuous scale which satisfies the UCT. Let  $B_T$  be as in 6.2. Suppose that there is  $\kappa \in KL(A, B_T)$ , an affine continuous map  $\kappa_T : T(A) \rightarrow T(B_T)$  and a continuous homomorphism  $\kappa_{uc} : U(\tilde{A})/CU(\tilde{A}) \rightarrow U(\tilde{B}_T)/CU(\tilde{B}_T)$  such that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible. Suppose also that  $\kappa|_{K_1(A)}$  is injective.*

*Then there exists a sequence of approximate multiplicative completely positive contractive linear maps  $\varphi_n : B_T \rightarrow A$  such that*

$$[\{\varphi_n\}] = \kappa, \quad (\text{e 11.73})$$

$$\lim_{n \rightarrow \infty} \sup\{|\tau \circ \varphi_n(a) - \kappa_T(\tau)(a)|\} = 0 \text{ for all } a \in (B_T)_{s.a.} \text{ and} \quad (\text{e 11.74})$$

$$\lim_{n \rightarrow \infty} \text{dist}(\kappa_{uc}(z), \varphi_n^\dagger(z)) = 0 \text{ for all } z \in U(\tilde{A})/CU(\tilde{A}). \quad (\text{e 11.75})$$

*Proof.* Denote by  $\Pi : U(\tilde{A})/CU(\tilde{A}) \rightarrow K_1(A)$  be the quotient map and fix a splitting map  $J_u : K_1(A) \rightarrow U(\tilde{A})/CU(\tilde{A})$ . Since  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible, it suffices to show that there are  $\{\varphi_n\}$  which satisfies (e 11.73) and (e 11.74) and

$$\lim_{n \rightarrow \infty} \text{dist}(\kappa_{uc}(J_u(\zeta)), \varphi_n^\dagger(J_u(\zeta))) = 0 \text{ for all } \zeta \in K_1(A). \quad (\text{e 11.76})$$

It follows from 11.6 that there exists  $\{\varphi_n\}$  which satisfies (e 11.73) and (e 11.74). Let  $G_1 \subset K_1(A)$  be a finitely generated subgroup.

Choose some sufficiently large  $n$ , then  $\varphi_n^\dagger$  induces a homomorphism on the  $J_u(G_1)$ . Since  $\kappa|_{K_1(A)}$  is injective and  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible,  $\varphi_n^\dagger|_{J_u(G_1)}$  has an inverse  $\gamma$ . Let  $G_b = \varphi_n^\dagger(J_u(G_1))$  and let  $\Pi_b : U(\tilde{B}_T)/CU(\tilde{B}_T) \rightarrow K_1(B_T)$  be the quotient map. Again, using the fact that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible,  $(\Pi_b)|_{G_b}$  is injective. Let  $J_{ub} : K_1(B_T) \rightarrow U(\tilde{B}_T)/CU(\tilde{B}_T)$  be a map such that  $\Pi_b \circ J_{ub} = \text{id}_{K_1(B_T)}$ .

Put

$$\lambda_0 = ((\kappa_{uc} \circ \gamma) \circ J_{uc} - (\varphi_n)^\dagger \circ \gamma \circ J_{uc})|_{\Pi_b(G_b)}. \quad (\text{e 11.77})$$

Then, since  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible,

$$\Pi_b \circ \lambda_0 = 0. \quad (\text{e 11.78})$$

Therefore  $\lambda_0$  maps from  $\Pi_b(G_b)$  to  $\overline{\text{Aff}(T(\tilde{B}_T))/\rho_{B_T}(K_1(\tilde{B}_T))}$ . However,  $\overline{\text{Aff}(T(\tilde{B}_T))/\rho_{B_T}(K_1(\tilde{B}_T))}$  is divisible. Therefore there is a homomorphism  $\lambda_1 : K_1(B_T) \rightarrow \overline{\text{Aff}(T(\tilde{B}_T))/\rho_{B_T}(K_1(\tilde{B}_T))}$  such that

$$(\lambda_1)|_{\Pi_b(G_b)} = \lambda_0. \quad (\text{e 11.79})$$



Now defined  $\Lambda : U(\tilde{B}_T)/CU(\tilde{B}_T) \rightarrow U(\tilde{B}_T)/CU(\tilde{B}_T)$  as follows.

$$\Lambda|_{\text{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}} = \text{id}_{\text{Aff}(T(\tilde{B}_T))/\overline{\rho_{B_T}(K_1(\tilde{B}_T))}}, \quad (\text{e 11.80})$$

$$\Lambda|_{J_{ub}(K_1(B_T))} = \lambda_1 \circ \Pi_b + (\text{id}_{B_T})^\dagger. \quad (\text{e 11.81})$$

Note that  $([\text{id}_{B_T}], (\text{id}_{B_T})_T, \Lambda)$  is compatible. It follows from 11.7 that there exists a homomorphism  $\psi_n : B_T \rightarrow B_T$  such that

$$[\psi_n] = [\text{id}_{B_T}], \quad (\psi_n)_T = (\text{id}_{B_T})_T \quad \text{and} \quad \psi_n^\dagger = \Lambda. \quad (\text{e 11.82})$$

Now let  $\Phi_n = \psi_n \circ \varphi_n$ . Then, for  $z \in J_u(G_1)$ , by (e 11.77),

$$\Phi_n^\dagger(z) = \psi_n^\dagger \circ \varphi_n^\dagger(z) = \lambda_1 \circ \Pi_b \circ \varphi_n^\dagger(z) + \varphi_n^\dagger(z) \quad (\text{e 11.83})$$

$$= \lambda_0 \circ \varphi_n^\dagger(z) + \varphi_n^\dagger(z) = \kappa_{uc}(z). \quad (\text{e 11.84})$$

The lemma follows immediately from this construction of  $\Phi_n$ . □

**Lemma 11.10.** *Let  $A$  be a non-unital simple separable  $C^*$ -algebra in  $\mathcal{D}_0$  with continuous scale which satisfies the UCT. Let  $B_T$  be as in 6.2. Suppose that there is  $\kappa \in KL(A, B_T)$ , an affine continuous map  $\kappa_T : T(A) \rightarrow T(B_T)$  and a continuous homomorphism  $\kappa_{uc} : U(\tilde{A})/CU(\tilde{A}) \rightarrow U(\tilde{B}_T)/CU(\tilde{B}_T)$  such that  $(\kappa, \kappa_T, \kappa_{uc})$  is compatible. Suppose also that  $\kappa|_{K_1(A)}$  is injective.*

*Then there exists a homomorphism  $\varphi : B_T \rightarrow A$  such that*

$$[\varphi] = \kappa, \quad \varphi_T = \kappa_T \quad \text{and} \quad \varphi^\dagger = \kappa_{uc}. \quad (\text{e 11.85})$$

*Proof.* The proof is exactly the same as that of 11.8 but applying 11.9 instead of 11.7. □

## 12 The Isomorphism Theorem for $\mathcal{Z}_0$ -stable $C^*$ -algebras

**Theorem 12.1.** *Let  $A$  and  $B$  be two separable simple amenable  $C^*$ -algebras in  $\mathcal{D}$  with continuous scale which satisfy the UCT. Suppose that  $\ker \rho_A = K_0(A)$  and  $\ker \rho_B = K_0(B)$ . Then  $A \cong B$  if and only if*

$$(K_0(A), K_1(A), T(A)) \cong (K_0(B), K_1(B), T(B)). \quad (\text{e 12.1})$$

*Moreover, let  $\kappa_i : K_i(A) \rightarrow K_i(B)$  be an isomorphism as abelian groups ( $i = 0, 1$ ) and let  $\kappa_T : T(B) \rightarrow T(A)$  be an affine homeomorphism. Suppose that  $\kappa \in KL(A, B)$  which gives  $\kappa_i$  and  $\kappa_{cu} : U(\tilde{A})/CU(\tilde{A}) \rightarrow U(\tilde{B})/CU(\tilde{B})$  be a continuous affine isomorphism so that  $(\kappa, \kappa_T, \kappa_{cu})$  is compatible. Then there is an isomorphism  $\varphi : A \rightarrow B$  such that*

$$[\varphi] = \kappa \quad (i = 0, 1, \quad \varphi_T = \kappa_T \quad \text{and} \quad \varphi^\dagger = \kappa_{cu}) \quad (\text{e 12.2})$$

*Proof.* Note it follows from 7.8 that  $A, B \in \mathcal{D}_0$ . It follows from 6.11 that there is a non-unital simple  $C^*$ -algebra  $B_T$  constructed in section 6 such that

$$K_0(B_T) = K_0(B), \quad K_1(B_T) = K_1(B) \quad \text{and} \quad T(B_T) = T(B). \quad (\text{e 12.3})$$

Let  $\kappa \in KL(A, B)$  be an invertible element which gives  $\kappa_i$  ( $i = 0, 1$ ). Let  $\kappa_T : T(B) \rightarrow T(A)$  be an affine homeomorphism. By the assumption,  $(\kappa, \kappa_T)$  is always compatible. Choose any  $\kappa_{cu}$  so that  $(\kappa, \kappa_T, \kappa_{cu})$ . Note that there is always at least one:  $\kappa_{cu}|_{J_c(K_1(A))} = J_c \circ \kappa|_{K_1(A)} \circ \pi_{cu}$ , where  $\pi_{cu} : U(\tilde{A})/CU(\tilde{A}) \rightarrow K_1(A)$  is the quotient map and  $\kappa_{cu}|_{\text{Aff}(T(A))/\mathbb{Z}}$  be induced by  $\kappa_T$ .

Therefore it suffices to show that there is an isomorphism  $\varphi : A \rightarrow B$  such that (e 12.2) holds. We will use the Elliott intertwining argument.

Let  $\{\mathcal{F}_{a,n}\}$  be an increasing sequence of finite subsets of  $A$  such that  $\cup_{n=1}^{\infty} \mathcal{F}_{a,n}$  is dense in  $A$ , let  $\{\mathcal{F}_{b,n}\}$  be an increasing sequence of finite subsets of  $B$  such that  $\cup_{n=1}^{\infty} \mathcal{F}_{b,n}$  is dense in  $B$ . Let  $\{\varepsilon_n\}$  be a sequence of decreasing positive numbers such that  $\sum_{n=1}^{\infty} \varepsilon_n < 1$ .

Let  $e_a \in A$  and  $e_b \in B$  be strictly positive elements of  $A$  and  $B$ , respectively, with  $\|e_a\| = 1$  and with  $\|e_b\| = 1$ . Note that  $d_{\tau}(e_a) = 1$  for all  $\tau \in T(A)$  and  $d_{\tau}(e_b) = 1$  for all  $\tau \in T(B)$ .

It follows from 11.10 that there is a homomorphism  $\varphi_1 : A \rightarrow B$  such that

$$[\varphi_1] = \kappa, \quad (\varphi_1)_T = \kappa_T \quad \text{and} \quad \varphi_1^{\dagger} = \kappa_{cu}. \quad (\text{e 12.4})$$

Note that  $d_{\tau}(\varphi_1(e_a)) = 1$ . Therefore  $\varphi_1$  maps  $e_a$  to a strictly positive element of  $B$ . It follows from 11.7 that there is a homomorphism  $\psi'_1 : B \rightarrow A$  such that

$$[\psi'_1] = \kappa^{-1}, \quad (\psi'_1)_T = \kappa_T^{-1} \quad \text{and} \quad (\psi'_1)^{\dagger} = \text{id}_A^{\dagger} \circ (\varphi_1^{\dagger})^{-1}. \quad (\text{e 12.5})$$

Thus

$$[\psi'_1 \circ \varphi_1] = [\text{id}_A], \quad (\psi'_1 \circ \varphi_1)_T = \text{id}_{T(A)} \quad \text{and} \quad (\psi'_1 \circ \varphi_1)^{\dagger} = \text{id}_{U(\tilde{A})/CU(\tilde{A})}. \quad (\text{e 12.6})$$

It follows from 5.3 (see also 5.7) that there exists a unitary  $u_{1,a} \in \tilde{A}$  such that

$$\text{Ad } u_{1,a} \circ \psi'_1 \circ \varphi_1 \approx_{\varepsilon_1} \text{id}_A \quad \text{on } \mathcal{F}_{a,1}. \quad (\text{e 12.7})$$

Put  $\psi_1 = \text{Ad } u_{1,a} \circ \psi'_1$ . Then we obtain the following diagram

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \varphi_1 \downarrow & \nearrow \psi_1 & \\ B & & \end{array}$$

which is approximately commutative on the subset  $\mathcal{F}_{a,1}$  within  $\varepsilon_1$ .

By applying 11.10, there exists a homomorphism  $\varphi'_2 : A \rightarrow B$  such that

$$[\varphi'_2] = \kappa, \quad (\varphi'_2)_T = \kappa_T \quad \text{and} \quad (\varphi'_2)^{\dagger} = \text{id}_B^{\dagger} \circ (\psi_1^{\dagger})^{-1} = \kappa_{cu}. \quad (\text{e 12.8})$$

Then,

$$[\varphi'_2 \circ \psi_1] = [\text{id}_B], \quad (\varphi'_2 \circ \psi_1)_T = \text{id}_{T(B)} \quad \text{and} \quad (\varphi'_2 \circ \psi_1)^{\dagger} = \text{id}_{U(\tilde{B})/CU(\tilde{B})}. \quad (\text{e 12.9})$$

It follows from 5.3 (and 5.7) that there exists a unitary  $u_{2,b} \in \tilde{B}$  such that

$$\text{Ad } u_{2,b} \circ \varphi'_2 \circ \psi_1 \approx_{\varepsilon_2} \text{id}_B \quad \text{on } \mathcal{F}_{b,2} \cup \varphi_1(\mathcal{F}_{a,1}). \quad (\text{e 12.10})$$

Put  $\varphi_2 = \text{Ad } u_{2,b} \circ \varphi'_2$ . Then we obtain the following diagram:

$$\begin{array}{ccc} A & \xrightarrow{\text{id}_A} & A \\ \varphi_1 \downarrow & \nearrow \psi_1 & \downarrow \varphi_2 \\ B & \xrightarrow{\text{id}_B} & B \end{array}$$

with the upper triangle approximately commutes on  $\mathcal{F}_{a,1}$  within  $\varepsilon_1$  and the lower triangle approximately commutes on  $\mathcal{F}_{b,2} \cup \varphi_1(\mathcal{F}_{a,1})$  within  $\varepsilon_2$ . Note also

$$[\varphi_2] = \kappa, \quad (\varphi_2)_T = \kappa_T \quad \text{and} \quad (\varphi_2)^{\dagger} = \kappa_{cu}. \quad (\text{e 12.11})$$

We then continue this process, and, by the induction, we obtain an approximate intertwining:

$$\begin{array}{ccccccc}
A & \xrightarrow{\text{id}_A} & A & \xrightarrow{\text{id}_A} & A & \xrightarrow{\text{id}_A} & \dots\dots A \\
\downarrow \varphi_1 & \nearrow \psi_1 & \downarrow \varphi_2 & \nearrow \psi_2 & \downarrow \varphi_3 & & \\
B & \xrightarrow{\text{id}_B} & B & \xrightarrow{\text{id}_B} & B & \xrightarrow{\text{id}_B} & \dots\dots B
\end{array}$$

By the Elliott approximate intertwining argument, this implies that  $A \cong B$  and the isomorphism  $\varphi$  produced by the above diagram meets the requirements of (e 12.2).  $\square$

The following theorem and its proof gives the proof of Theorem 1.1.

**Theorem 12.2.** *Let  $A$  and  $B$  be two stably projectionless separable simple amenable  $C^*$ -algebras with  $gTR(A) \leq 1$  and  $gTR(B) \leq 1$  and which satisfy the UCT. Suppose that  $K_0(A) = \ker \rho_A$  and  $K_0(B) = \ker \rho_B$ . Then  $A \cong B$  if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \quad (\text{e 12.12})$$

*Proof.* Let

$$\Gamma : (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \rightarrow (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B) \quad (\text{e 12.13})$$

be an isomorphism. Let  $\Gamma_T : \tilde{T}(A) \rightarrow \tilde{T}(B)$  be the cone homeomorphism such that

$$\Sigma_B(\Gamma_T(\tau)) = \Sigma_A(\tau) \text{ for all } \tau \in \tilde{T}(A). \quad (\text{e 12.14})$$

Let  $e_A \in P(A)_+$  such that  $\|e_A\| = 1$  such that  $A_0 := \overline{e_A A e_A}$  has continuous scale (see 2.3 of [28]). Choose  $b_0 \in P(B)_+ \setminus \{0\}$  with  $\|b_0\| = 1$  such that  $B' := \overline{b_0 B b_0}$  has continuous scale. Then  $T(A_0)$  and  $T(B')$  are metrizable Choquet simplexes. Moreover  $T(A_0)$  and  $T(B')$  can be identified with

$$T_A = \{\tau \in \tilde{T}(A) : d_\tau(a_A) = 1\} \text{ and } \{s \in \tilde{T}(B') : d_s(b_0) = 1\}, \quad (\text{e 12.15})$$

respectively. Let  $g(t) = d_{\Gamma^{-1}(t)}(e_A) \in \text{LAff}_f(\tilde{T}(B))$ . Since  $d_\tau(e_A)$  is continuous and  $\Gamma^{-1}$  is a cone homeomorphism,  $g(t)$  is continuous and  $g \in \text{Aff}_+(T(B'))$ . Since  $\text{Aff}_+(T(B'))$  is compact,  $g$  is also bounded. By identifying  $B' \otimes \mathcal{K}$  with  $B \otimes \mathcal{K}$ , we find a positive element  $b_{00} = \text{diag}(b_0, \dots, b_0) \in B \otimes \mathcal{K}$ , where  $b_0$  repeats  $m$  times so that  $d_s(b_{00}) > g(s)$  on  $T(B')$ . Then  $g$  is continuous on  $T(B'')$ , where  $B'' := \overline{b_{00}}(B \otimes \mathcal{K})b_{00}$ . It follows 7.6 that there is  $e_B \in B''_+ \subset B \otimes \mathcal{K}$  with  $\|e_B\| = 1$  such that  $d_s(e_B) = g|_{T(B'')}$ . Since  $B$  has strictly comparison,  $B_0 := \overline{e_B B e_B}$  has continuous scale (see 9.3 of [17]). Let

$$T_B = \{t \in \tilde{T}(B) : d_t(e_B) = 1\}. \quad (\text{e 12.16})$$

Then  $T(A_0) = T_B$ . It follows that  $\Gamma$  induces the following isomorphism

$$(K_0(A_0), K_1(A_0), T(A_0)) \cong (K_0(B_0), K_1(B_0), T(B_0)). \quad (\text{e 12.17})$$

It follows from 12.1 that there is an isomorphism  $\varphi_0 : A_0 \rightarrow B_0$  which induces  $\Gamma$  on  $(K_0(A_0), K_1(A_0), T(A_0))$ . By [3],  $\varphi_0$  gives an isomorphism from  $A_0 \otimes \mathcal{K}$  onto  $B_0 \otimes \mathcal{K}$ . Let  $a \in A_+$  with  $\|a\| = 1$  be a strictly positive element. Then

$$\hat{a}(\tau) = \Sigma_A(\tau) \text{ for all } \tau \in \tilde{T}(A). \quad (\text{e 12.18})$$

Let  $b \in (B_0 \otimes \mathcal{K})_+$  such that  $\varphi(a) = b$ . Then

$$d_t(b) = \lim_{n \rightarrow \infty} t \circ \varphi(a^{1/n}) \text{ for all } t \in \tilde{T}(B). \quad (\text{e 12.19})$$

Note  $\Sigma_B(t) = d_t(b)$ . Since  $B$  is simple and has stable rank one, this implies that  $B \cong \overline{b(B_0 \otimes \mathcal{K})b}$ . The theorem follows.  $\square$

**Corollary 12.3.** *Let  $A$  and  $B$  be in  $\mathcal{D}_0$  which are amenable and satisfy the UCT. Then  $A \cong B$  if and only if*

$$\text{Ell}(A) \cong \text{Ell}(B). \quad (\text{e 12.20})$$

*Proof.* Since  $A$  and  $B$  are in  $\mathcal{D}_0$ , by 7.5,  $K_0(A) = \ker \rho_A$  and  $K_0(B) = \ker \rho_B$ . Therefore Theorem 12.2 applies.  $\square$

**Corollary 12.4.** *Let  $A$  be a stably projectionless simple separable amenable  $C^*$ -algebra which satisfies the UCT and  $gTR(A) \leq 1$ . Suppose that  $K_0(A) = \ker \rho_A$ . Then  $A \otimes \mathcal{Z}_0 \cong A$ .*

*In particular,  $\mathcal{Z}_0 \otimes \mathcal{Z}_0 \cong \mathcal{Z}_0$ .*

*Proof.* Recall that  $K_0(\mathcal{Z}_0) = \mathbb{Z} = \ker \rho_{\mathcal{Z}_0}$ ,  $K_1(\mathcal{Z}_0) = \{0\}$  and  $T(\mathcal{Z}_0)$  has exactly one point. By 18.5 and 18.6 of [17],  $gTR(A \otimes \mathcal{Z}_0) \leq 1$ . In particular, by 18.5 of [17], every tracial state of  $T(eAe)$  is a W-trace (for any  $e \in P(A)_+ \setminus \{0\}$ ). It follows from 18.3 of [17] that  $K_0(A \otimes \mathcal{Z}_0) = \ker \rho_A$ . Moreover,  $K_0(A \otimes \mathcal{Z}_0) \cong K_0(A) = \ker \rho_A$ ,  $K_1(A \otimes \mathcal{Z}_0) \cong K_1(A)$  and  $\tilde{T}(A \otimes \mathcal{Z}_0) = \tilde{T}(A)$ . Thus 12.2 applies.  $\square$

### 13 A homotopy lemma

The purpose of this section is to present 13.10 which will be used in next section. The following is known, a proof for the unital case can be found in 12.4 of [18]

**Lemma 13.1.** *Let  $C$  be a separable  $C^*$ -algebra, and let  $\Delta : C_+^{q,1} \setminus \{0\} \rightarrow (0,1)$  be an order preserving map. There exists a map  $T : C_+ \setminus \{0\} \rightarrow \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$  satisfying the following: For any finite subset  $\mathcal{H} \subset C_+^1 \setminus \{0\}$  and any  $\sigma$ -unital  $C^*$ -algebra  $A$  with the strict comparison of positive elements which is quasi-compact, if  $\varphi : C \rightarrow A$  is a unital contractive completely positive linear map satisfying*

$$\tau \circ \varphi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H} \text{ for all } \tau \in T(A), \quad (\text{e 13.1})$$

*then  $\varphi$  is  $T$ - $\mathcal{H}$ -full.*

**Theorem 13.2.** *Let  $A_0$  be a non-unital  $C^*$ -algebra such that  $A := \tilde{A}_0 \in \overline{\mathcal{D}}_s$  (for some  $s \geq 1$  — see 4.8 of [18]) with finitely generated  $K_i(A)$  ( $i = 0, 1$ ). Let  $\mathcal{F} \subset A$  be a finite subset, let  $\varepsilon > 0$  be a positive number and let  $\Delta : A_+^{q,1} \setminus \{0\} \rightarrow (0,1)$  be an order preserving map. There exists a finite subset  $\mathcal{H}_1 \subset A_+^1 \setminus \{0\}$ ,  $\gamma_1 > 0$ ,  $\gamma_2 > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset A$  and a finite subset  $\mathcal{P} \subset \underline{K}(A)$ , a finite subset  $\mathcal{H}_2 \subset A$ , a finite subset  $\mathcal{U} \subset J_c(K_1(A))$  (see (e 2.9) in 2.4 for the definition of  $J_c$ .) for which  $[\mathcal{U}] \subset \mathcal{P}$  satisfying the following: For any unital  $\mathcal{G}$ - $\delta$ -multiplicative contractive completely positive linear maps  $\varphi, \psi : A_0 \rightarrow C$  for some  $C \in \mathcal{C}_0$  such that*

$$[\varphi^\sim]_{\mathcal{P}} = [\psi^\sim]_{\mathcal{P}}, \quad (\text{e 13.2})$$

$$\tau(\varphi^\sim(a)) \geq \Delta(\hat{a}), \quad \tau(\psi^\sim(a)) \geq \Delta(\hat{a}), \quad \text{for all } \tau \in T(C) \text{ and } a \in \mathcal{H}_1, \quad (\text{e 13.3})$$

$$|\tau \circ \varphi^\sim(a) - \tau \circ \psi^\sim(a)| < \gamma_1 \text{ for all } a \in \mathcal{H}_2, \text{ and} \quad (\text{e 13.4})$$

$$\text{dist}((\varphi^\sim)^\dagger(u), (\psi^\sim)^\dagger(u)) < \gamma_2 \text{ for all } u \in \mathcal{U}, \quad (\text{e 13.5})$$

*there exists a unitary  $W \in \tilde{C}$  such that*

$$\|W(\varphi^\sim(f))W^* - (\psi^\sim(f))\| < \varepsilon, \text{ for all } f \in \mathcal{F}, \quad (\text{e 13.6})$$

*where  $\varphi^\sim, \psi^\sim$  are the unital extension of  $\varphi$  and  $\psi$  from  $A$  to  $\tilde{C}$ .*

*Proof.* Without loss of generality, we may assume that  $A$  is infinite dimensional.

Since  $K_*(A)$  is finitely generated, there is  $n_0$  such that  $\kappa \in \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(C))$  is determined by its restriction to  $K(A, \mathbb{Z}/n\mathbb{Z})$ ,  $n = 0, \dots, n_0$ .

Let  $\mathcal{H}'_1 \subset A_+ \setminus \{0\}$  (in place of  $\mathcal{H}_1$ ) for  $\varepsilon/32$  (in place of  $\varepsilon$ ) and  $\mathcal{F}$  required by 6.7 of [18].

Let  $\delta_1 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_1 \subset A$  (in place of  $\mathcal{G}$ ) be a finite subset and let  $\mathcal{P}_0 \subset \underline{K}(A)$  (in place of  $\mathcal{P}$ ) be a finite subset required by 6.7 of [18] for  $\varepsilon/32$  (in place of  $\varepsilon$ ),  $\mathcal{F}$  and  $\Delta$ . We may assume that  $\delta_1 < \varepsilon/32$  and  $(2\delta_1, \mathcal{G}_1)$  is a  $KK$ -pair (see the end of 2.12 of [18]).

Moreover, we may assume that  $\delta_1$  is sufficiently small that if  $\|uv - vu\| < 3\delta_1$ , then the Exel formula

$$\tau(\text{bott}_1(u, v)) = \frac{1}{2\pi\sqrt{-1}}(\tau(\log(u^*vu v^*)))$$

holds for any pair of unitaries  $u$  and  $v$  in any unital  $C^*$ -algebra  $C$  with tracial rank zero and any  $\tau \in T(C)$  (see Theorem 3.6 of [30]). Moreover if  $\|v_1 - v_2\| < 3\delta_1$ , then

$$\text{bott}_1(u, v_1) = \text{bott}_1(u, v_2).$$

Let  $g_1, g_2, \dots, g_{k(A)} \in U(M_{m(A)}(A))$  ( $m(A) \geq 1$  is an integer) be a finite subset such that  $\{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{k(A)}\} \subset J_c(K_1(A))$  and such that  $\{[g_1], [g_2], \dots, [g_{k(A)}]\}$  forms a set of generators for  $K_1(A)$ . Let  $\mathcal{U} = \{\bar{g}_1, \bar{g}_2, \dots, \bar{g}_{k(A)}\} \subset J_c(K_1(A))$  be a finite subset.

Let  $\mathcal{U}_0 \subset A$  be a finite subset such that

$$\{g_1, g_2, \dots, g_{k(A)}\} \subseteq \{(a_{i,j}) : a_{i,j} \in \mathcal{U}_0\}.$$

Let  $\delta_u = \min\{1/256m(A)^2, \delta_1/16m(A)^2\}$ ,  $\mathcal{G}_u = \mathcal{F} \cup \mathcal{G}_1 \cup \mathcal{U}_0$  and let  $\mathcal{P}_u = \mathcal{P}_0 \cup \{[g_1], [g_2], \dots, [g_{k(A)}]\}$ .

Let  $\delta_2 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_2 \subset A$  (in place of  $\mathcal{G}$ ),  $\mathcal{H}'_2 \subset A_+ \setminus \{0\}$  (in place of  $\mathcal{H}$ ),  $N_1 \geq 1$  (in place of  $N$ ) be the finite subsets and the constants as required by 7.4 of [18] for  $\delta_u$  (in place of  $\varepsilon$ ),  $\mathcal{G}_u$  (in place of  $\mathcal{F}$ ),  $\mathcal{P}_u$  (in place of  $\mathcal{P}$ ) and  $\Delta$  and with  $\bar{g}_j$  (in place of  $g_j$ ),  $j = 1, 2, \dots, k(A)$  (with  $k(A) = r$ ).

Let  $\delta_3 > 0$  and let  $\mathcal{G}_3 \subset A \otimes C(\mathbb{T})$  be a finite subset satisfying the following: For any  $\mathcal{G}_3$ - $\delta_3$ -multiplicative contractive completely positive linear map  $L' : A \otimes C(\mathbb{T}) \rightarrow C'$  (for any unital  $C^*$ -algebra  $C'$  with  $T(C') \neq \emptyset$ ),

$$|\tau([L'](\beta(\bar{g}_j)))| < 1/8N_1, \quad j = 1, 2, \dots, k(A). \quad (\text{e 13.7})$$

Without loss of generality, we may assume that

$$\mathcal{G}_3 = \{g \otimes f : g \in \mathcal{G}'_3 \text{ and } f \in \{1, z, z^*\}\},$$

where  $\mathcal{G}'_3 \subset A$  is a finite subset containing  $1_A$  (by choosing a smaller  $\delta_3$  and large  $\mathcal{G}'_3$ ).

Let  $\varepsilon'_1 = \min\{d/27N_1m(A)^2, \delta_u/2, \delta_2/2m(A)^2, \delta_3/2m(A)^2\}$  and let  $\varepsilon_1 > 0$  (in place of  $\delta$ ) and  $\mathcal{G}_4 \subset A$  (in place of  $\mathcal{G}$ ) be a finite subset as required by 6.4 of [18] for  $\varepsilon'_1$  (in place of  $\varepsilon$ ) and  $\mathcal{G}_u \cup \mathcal{G}'_3$ . Put  $\varepsilon_1 = \min\{\varepsilon'_1, \varepsilon''_1, \bar{\varepsilon}_1\}$ . Let  $\mathcal{G}_5 = \mathcal{G}_u \cup \mathcal{G}'_3 \cup \mathcal{G}_4$ .

Let  $\mathcal{H}'_3 \subseteq A_+^1 \setminus \{0\}$  (in place of  $\mathcal{H}_1$ ),  $\delta_4 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_6 \subset A$  (in place of  $\mathcal{G}$ ),  $\mathcal{H}'_4 \subset A_{s.a.}$  (in place of  $\mathcal{H}_2$ ),  $\mathcal{P}_1 \subset \underline{K}(A)$  (in place of  $\mathcal{P}$ ) and  $\sigma > 0$  be the finite subsets and constants as required by 5.9 of [18] with respect to  $\varepsilon_1/4$  (in place of  $\varepsilon$ ) and  $\mathcal{G}_5$  (in place of  $\mathcal{F}$ ) and  $\Delta$ .

Choose  $N_2 \geq N_1$  such that  $(k(A) + 1)/N_2 < 1/8N_1$ . Choose  $\mathcal{H}'_5 \subset A_+^1 \setminus \{0\}$  and  $\delta_5 > 0$  and a finite subset  $\mathcal{G}_7 \subset A$  such that, for any  $M_m$  and unital  $\mathcal{G}_7$ - $\delta_5$ -multiplicative contractive completely positive linear map  $L' : A \rightarrow M_m$ , if  $\text{tr} \circ L'(h) > 0$  for all  $h \in \mathcal{H}'_5$ , then  $m \geq 16N_2$ .

Put  $\delta = \min\{\varepsilon_1/16, \delta_4/4m(A)^2, \delta_5/4m(A)^2\}$ ,  $\mathcal{G} = \mathcal{G}_5 \cup \mathcal{G}_6 \cup \mathcal{G}_7$ , and  $\mathcal{P} = \mathcal{P}_u \cup \mathcal{P}_1$ . Put

$$\mathcal{H}_1 = \mathcal{H}'_1 \cup \mathcal{H}'_2 \cup \mathcal{H}'_3 \cup \mathcal{H}'_4 \cup \mathcal{H}'_5$$

and let  $\mathcal{H}_2 = \mathcal{H}'_4$ . Let  $\gamma_1 = \sigma$  and let  $0 < \gamma_2 < \min\{d/16N_2m(A)^2, \delta_u/9m(A)^2, 1/256m(A)^2\}$ .

Now suppose that  $C \in \mathcal{C}_0$  and  $\varphi, \psi : A \rightarrow C$  are two unital  $\mathcal{G}$ - $\delta$ -multiplicative contractive completely positive linear maps satisfying the condition of the theorem for the given  $\Delta, \mathcal{H}_1, \delta, \mathcal{G}, \mathcal{P}, \mathcal{H}_2, \gamma_1, \gamma_2$  and  $\mathcal{U}$ .

We write  $C = A(F_1, F_2, h_0, h_1)$ ,  $F_1 = M_{m_1} \oplus M_{m_2} \oplus \cdots \oplus M_{m_{F(1)}}$  and  $F_2 = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_{F(2)}}$ . By the choice of  $\mathcal{H}'_5$ , one has that

$$n_j \geq 16N_2 \text{ and } m_s \geq 16N_2, \quad 1 \leq j \leq F(2), \quad 1 \leq s \leq F(1). \quad (\text{e } 13.8)$$

Let  $q_{F_1,0} = h_0(1_{F_1})$  and  $q_{F_1,1} = h_1(1_{F_1})$ . Define  $h_0^\sim : F_1 \oplus \mathbb{C} \rightarrow F_2$  by  $h_0^\sim((a, \lambda)) = h_0(a) \oplus \lambda(1 - q_{F_1,0})$  and  $h_1^\sim((a, \lambda)) = h_1(a) \oplus \lambda(1 - q_{F_1,1})$ . Then  $\tilde{C} = A(F_1 \oplus \mathbb{C}, F_2, h_0^\sim, h_1^\sim)$ . Put  $\pi^{C^\sim} : \tilde{C} \rightarrow \mathbb{C}$ . Then

$$\pi^{C^\sim} \circ \varphi(a) = \pi^{C^\sim} \circ \psi(a) \text{ for all } a \in A. \quad (\text{e } 13.9)$$

Let  $0 = t_0 < t_1 < \cdots < t_n = 1$  be a partition of  $[0, 1]$  so that

$$\|\pi_t \circ \varphi^\sim(g) - \pi_{t'} \circ \varphi^\sim(g)\| < \varepsilon_1/16 \text{ and } \|\pi_t \circ \psi^\sim(g) - \pi_{t'} \circ \psi^\sim(g)\| < \varepsilon_1/16 \quad (\text{e } 13.10)$$

for all  $g \in \mathcal{G}$ , provided  $t, t' \in [t_{i-1}, t_i]$ ,  $i = 1, 2, \dots, n$ .

Applying Theorem 5.9 of [18], one obtains a unitary  $w_i \in F_2$  if  $0 < i < n$ ,  $w_0 \in h_0(F_1)$ , such that

$$\|w_i \pi_{t_i} \circ \varphi(g) w_i^* - \pi_{t_i} \circ \psi(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5, \quad (\text{e } 13.11)$$

Also there is  $w'_e \in F_1$  such that

$$\|(w'_e)^* \pi_e \circ \varphi(g) w'_e - \pi_e \circ \psi(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5. \quad (\text{e } 13.12)$$

Let  $\pi^{F_1'^\sim} : h_0^\sim(F_1^\sim) \rightarrow \mathbb{C}$  and let  $\pi' : h_0(F_1^\sim) \rightarrow h_0(F_1)$  be the quotient maps. Put  $w_0 = h_0(w'_e) \oplus (1_{F_2} - q_{F_1,0})$ ,  $w_n = h_1(w'_e) \oplus (1_{F_2} - q_{F_1,1})$ ,  $w'_0 = h_0(w'_e)$  and  $w'_n = h_1(w'_e)$ . Then

$$\|w_i^* \pi_i \circ \varphi^\sim(g) w_i - \pi_i \circ \psi^\sim(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_5. \quad (\text{e } 13.13)$$

$i = 0$  and  $i = n$ . By (e 13.5), there is a unitary  $\omega_j \in M_{m(A)}(\tilde{C})$  such that  $\omega_j \in CU(M_{m(A)}(\tilde{C}))$  and

$$\|[(\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j^*)][(\psi^\sim \otimes \text{id}_{M_{m(A)}})(g_j)] - \omega_j\| < \gamma_2, \quad j = 1, 2, \dots, k(A). \quad (\text{e } 13.14)$$

(note that we now have  $w_i$  as well as  $\omega_i$  in the proof.) Write

$$\omega_j = \prod_{l=1}^{e(j)} \exp(\sqrt{-1} a_j^{(l)})$$

for some selfadjoint element  $a_j^{(l)} \in M_{m(A)}(\tilde{C})$ ,  $l = 1, 2, \dots, e(j)$ ,  $j = 1, 2, \dots, k(A)$ . Write

$$a_j^{(l)} = (a_j^{(l,1)}, a_j^{(l,2)}, \dots, a_j^{(l,n_{F(2)})}) \text{ and } \omega_j = (\omega_{j,1}, \omega_{j,2}, \dots, \omega_{j,F(2)})$$

in  $C([0, 1], F_2) = C([0, 1], M_{n_1}) \oplus \cdots \oplus C([0, 1], M_{n_{F(2)}})$ , where  $\omega_{j,s} = \prod_{l=1}^{e(j)} \exp(\sqrt{-1} a_j^{(l,s)})$ ,  $s = 1, 2, \dots, F(2)$ . Let  $r_{l,j,s} = \pi^{C^\sim}(a_j^{(l,s)})$ ,  $l = 1, 2, \dots, e(1)$ . Then  $r_{l,j,s} \in \mathbb{R}$ . Then  $\sum_{l=1}^{e(j)} r_{l,j,s} \in 2\pi\mathbb{Z}$ . Replacing  $a_j^{(l,s)}$  by  $a_j^{(l,s)} - r_{l,j,s}$ ,  $l = 1, 2, \dots, e(j)$ , we may assume that  $\pi^{C^\sim}(a_j^{(l,s)}) = 0$ ,  $l = 1, 2, \dots, e(j)$ ,  $j = 1, 2, \dots, k(A)$ .

Then

$$\sum_{l=1}^{e(j)} \frac{n_s(t_s \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t))}{2\pi} \in \mathbb{Z}, \quad t \in (0, 1),$$

where  $t_s$  is the normalized trace on  $M_{n_s}$ ,  $s = 1, 2, \dots, F(2)$ . In particular,

$$\sum_{l=1}^{e(j)} n_s(t_s \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t)) = \sum_{l=1}^{e(j)} n_s(t_s \otimes \text{Tr}_{m(A)})(a_j^{(l,s)}(t')) \text{ for all } t, t' \in (0, 1). \quad (\text{e 13.15})$$

We also have

$$(1/2\pi) \sum_{l=1}^{e(j)} m_s(t_{es} \otimes \text{Tr}_{m(A)})(\pi_e(a_j^{(l,s)})) \in \mathbb{Z}, \quad (\text{e 13.16})$$

where  $t_{es}$  is the tracial state on  $M_{m_s}$ .

Let  $W_i = w_i \otimes \text{id}_{M_{m(A)}}$ ,  $i = 0, 1, \dots, n$  and  $W_e = w_e \otimes \text{id}_{M_{m(A)}(F_1)}$ . Then

$$\|\pi_{t_i}([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j^*)]W_i(\pi_{t_i}([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j)])W_i^* - \omega_j(t_i)\| \quad (\text{e 13.17})$$

$$< 3m(A)^2 \varepsilon_1 + 2\gamma_2 < 1/32. \quad (\text{e 13.18})$$

We also have (with  $\varphi_e = \pi_e \circ \varphi^\sim$ )

$$\|[(\varphi_e \otimes \text{id}_{M_{m(A)}})(g_j^*)]W_e([\varphi_e \otimes \text{id}_{M_{m(A)}})(g_j)])W_e^* - \pi_e(\omega_j)\| < 3m(A)^2 \varepsilon_1 + 2\gamma_2 < 1/32. \quad (\text{e 13.19})$$

It follows from (e 13.17) that there exists selfadjoint elements  $b_{i,j} \in M_{m(A)}(F_2)$  such that

$$\exp(\sqrt{-1}b_{i,j}) = \omega_j(t_i)^*(\pi_i([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j^*)]W_i(\pi_i([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j)])W_i^*, \quad (\text{e 13.20})$$

and  $b_{e,j} \in M_{m(A)}(F_1)$  such that

$$\exp(\sqrt{-1}b_{e,j}) = \pi_e(\omega_j)^*(\pi_e([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j^*)]W_e(\pi_e([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j)])W_e^*, \quad (\text{e 13.21})$$

and

$$\|b_{i,j}\| < 2 \arcsin(3m(A)^2 \varepsilon_1/2 + \gamma_2), \quad j = 1, 2, \dots, k(A), \quad i = 0, 1, \dots, n, e. \quad (\text{e 13.22})$$

Write

$$b_{i,j} = (b_{i,j}^{(1)}, b_{i,j}^{(2)}, \dots, b_{i,j}^{(F(2))}) \in F_2 \text{ and } b_{e,j} = (b_{e,j}^{(1)}, b_{e,j}^{(2)}, \dots, b_{e,j}^{(F(1))}) \in F_1.$$

We have that

$$h_0(b_{e,j}) = b_{0,j} \text{ and } h_1(b_{e,j}) = b_{n,j}. \quad (\text{e 13.23})$$

Note that

$$(\pi_{t_i}([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j^*)]W_i(\pi_{t_i}([\varphi^\sim \otimes \text{id}_{M_{m(A)}})(g_j)])W_i^* = \pi_{t_i}(\omega_j) \exp(\sqrt{-1}b_{i,j}), \quad (\text{e 13.24})$$

$j = 1, 2, \dots, k(A)$  and  $i = 0, 1, \dots, n, e$ . Then,

$$\frac{n_s}{2\pi}(t_s \otimes \text{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z}, \quad (\text{e 13.25})$$

where  $t_s$  is the normalized trace on  $M_{n_s}$ ,  $s = 1, 2, \dots, F(2)$ ,  $j = 1, 2, \dots, k(A)$ , and  $i = 0, 1, \dots, n$ .

We also have

$$\frac{m_s}{2\pi}(t_s \otimes \text{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z}, \quad (\text{e 13.26})$$



where  $t_s$  is the normalized trace on  $M_{m_s}$ ,  $s = 1, 2, \dots, F(1)$ ,  $j = 1, 2, \dots, k(A)$ . Put

$$\lambda_{i,j}^{(s)} = \frac{n_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}})(b_{i,j}^{(s)}) \in \mathbb{Z},$$

where  $t_s$  is the normalized trace on  $M_{n_s}$ ,  $s = 1, 2, \dots, n$ ,  $j = 1, 2, \dots, k(A)$  and  $i = 0, 1, 2, \dots, n$ .

Put

$$\lambda_{e,j}^{(s)} = \frac{m_s}{2\pi} (t_s \otimes \text{Tr}_{M_{m(A)}})(b_{e,j}^{(s)}) \in \mathbb{Z},$$

where  $t_s$  is the normalized trace on  $M_{m_s}$ ,  $s = 1, 2, \dots, F(1)$  and  $j = 1, 2, \dots, k(A)$ . Denote

$$\lambda_{i,j} = (\lambda_{i,j}^{(1)}, \lambda_{i,j}^{(2)}, \dots, \lambda_{i,j}^{(F(2))}) \in \mathbb{Z}^{F(2)}, \text{ and } \lambda_{e,j} = (\lambda_{e,j}^{(1)}, \lambda_{e,j}^{(2)}, \dots, \lambda_{e,j}^{(F(1))}) \in \mathbb{Z}^{F(1)}.$$

We have, by (e 13.22), for  $j = 1, 2, \dots, k(A)$  and  $i = 0, 1, 2, \dots, n$ ,

$$\left| \frac{\lambda_{i,j}^{(s)}}{n_s} \right| < 1/4N_1, \quad s = 1, 2, \dots, F(2), \quad \left| \frac{\lambda_{e,j}^{(s)}}{m_s} \right| < 1/4N_1, \quad s = 1, 2, \dots, F(1), \quad (\text{e 13.27})$$

Define  $\alpha_i^{(0,1)} : K_1(A) \rightarrow \mathbb{Z}^{F(2)}$  by mapping  $[g_j]$  to  $\lambda_{i,j}$ ,  $j = 1, 2, \dots, k(A)$ ,  $i = 0, 1, 2, \dots, n$ , and define  $\alpha_e^{(0,1)} : K_1(A) \rightarrow \mathbb{Z}^{F(1)} \oplus \mathbb{Z}$  by mapping  $[g_j]$  to  $(\lambda_{e,j}, 0)$ ,  $j = 1, 2, \dots, k(A)$ . We write  $K_0(A \otimes C(\mathbb{T})) = K_0(A) \oplus \beta(K_1(A))$  (see 2.10 of [31] for the definition of  $\beta$ ). Define  $\alpha_i : K_*(A \otimes C(\mathbb{T})) \rightarrow K_*(F_2)$  as follows: On  $K_0(A \otimes C(\mathbb{T}))$ , define

$$\alpha_i|_{K_0(A)} = [\pi_i \circ \varphi]|_{K_0(A)}, \quad \alpha_i|_{\beta(K_1(A))} = \alpha_i \circ \beta|_{K_1(A)} = \alpha_i^{(0,1)} \quad (\text{e 13.28})$$

and on  $K_1(A \otimes C(\mathbb{T}))$ , define  $\alpha_i|_{K_1(A \otimes C(\mathbb{T}))} = 0$ ,  $i = 0, 1, 2, \dots, n$ .

Also define  $\alpha_e \in \text{Hom}(K_*(A \otimes C(\mathbb{T})), K_*(F_1 \otimes \mathbb{C}))$ , by

$$\alpha_e|_{K_0(A)} = [\pi_e \circ \varphi^\sim]|_{K_0(A)}, \quad \alpha_e|_{\beta(K_1(A))} = \alpha_e \circ \beta|_{K_1(A)} = \alpha_e^{(0,1)} \quad (\text{e 13.29})$$

on  $K_0(A \otimes C(\mathbb{T}))$  and  $(\alpha_e)|_{K_1(A \otimes C(\mathbb{T}))} = 0$ . Note that

$$(h_0^\sim)_* \circ \alpha_e = \alpha_0 \text{ and } (h_1^\sim)_* \circ \alpha_e = \alpha_n. \quad (\text{e 13.30})$$

Since  $A \otimes C(\mathbb{T})$  satisfies the UCT, the map  $\alpha_e$  can be lifted to an element of  $KK(A \otimes C(\mathbb{T}), F_1 \oplus \mathbb{C})$  which is still denoted by  $\alpha_e$ . Then define

$$\alpha_0 = \alpha_e \times [h_0^\sim] \text{ and } \alpha_n = \alpha_e \times [h_1^\sim] \quad (\text{e 13.31})$$

in  $KK(A \otimes C(\mathbb{T}), F_2)$ . For  $i = 1, \dots, n-1$ , also pick a lifting of  $\alpha_i$  in  $KK(A \otimes C(\mathbb{T}), F_2)$ , and still denote it by  $\alpha_i$ . We estimate that

$$\|(w_i^* w_{i+1}) \pi_{t_i} \circ \varphi^\sim(g) - \pi_{t_i} \circ \varphi^\sim(g) (w_i^* w_{i+1})\| < \varepsilon_1/4 \text{ for all } g \in \mathcal{G}_5, \quad (\text{e 13.32})$$

$i = 0, 1, \dots, n-1$ . Let  $\Lambda_{i,i+1} : C(\mathbb{T}) \otimes A \rightarrow F_2$  be a unital contractive completely positive linear map given by the pair  $w_i^* w_{i+1}$  and  $\pi_{t_i} \circ \varphi$  (by 6.4 of [18], see 2.8 of [31]). Denote  $V_{i,j} = [\pi_{t_i} \circ \varphi^\sim \otimes \text{id}_{M_{m(A)}}](g_j)$ ,  $j = 1, 2, \dots, k(A)$  and  $i = 0, 1, 2, \dots, n-1$ .

Write

$$V_{i,j} = (V_{i,j,1}, V_{i,j,2}, \dots, V_{i,j,F(2)}) \in M_{m(A)}(F_2), \quad j = 1, 2, \dots, k(A), \quad i = 0, 1, 2, \dots, n.$$

Similarly, write

$$W_i = (W_{i,1}, W_{i,2}, \dots, W_{i,F(2)}) \in M_{m(A)}(F_2), \quad i = 0, 1, 2, \dots, n. \quad (\text{e 13.33})$$

We have

$$\|W_i V_{i,j}^* W_i^* V_{i,j} V_{i,j}^* W_{i+1} V_{i,j} W_{i+1}^* - 1\| < 1/16 \quad (\text{e 13.34})$$

$$\|W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* - 1\| < 1/16 \quad (\text{e 13.35})$$

and there is a continuous path  $Z(t)$  of unitaries such that  $Z(0) = V_{i,j}$  and  $Z(1) = V_{i+1,j}$ . Since

$$\|V_{i,j} - V_{i+1,j}\| < \delta_1/12, \quad j = 1, 2, \dots, k(A),$$

we may assume that  $\|Z(t) - Z(1)\| < \delta_1/6$  for all  $t \in [0, 1]$ . We also write

$$Z(t) = (Z_1(t), Z_2(t), \dots, Z_{F(2)}(t)) \in F_2 \quad \text{and} \quad t \in [0, 1].$$

We obtain a continuous path  $W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^*$  which is in  $CU(M_{nm(A)})$  for all  $t \in [0, 1]$  and

$$\|W_i V_{i,j}^* W_i^* V_{i,j} Z(t)^* W_{i+1} Z(t) W_{i+1}^* - 1\| < 1/8 \quad \text{for all } t \in [0, 1].$$

It follows that

$$(1/2\pi\sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}})[\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* V_{i,j,s} Z_s(t)^* W_{i+1,s} Z_s(t) W_{i+1,s}^*)]$$

is a constant integer, where  $t_s$  is the normalized trace on  $M_{n_s}$ . In particular,

$$(1/2\pi\sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}})(\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* W_{i+1,s} V_{i,j,s} W_{i+1,s}^*)) \quad (\text{e 13.36})$$

$$= (1/2\pi\sqrt{-1})(t_s \otimes \text{Tr}_{M_{m(A)}})(\log(W_{i,s} V_{i,j,s}^* W_{i,s}^* V_{i,j} V_{i+1,j}^* W_{i+1,s} V_{i,j,s} W_{i+1,s}^*)). \quad (\text{e 13.37})$$

One also has

$$W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^* = (\omega_j(t_i) \exp(\sqrt{-1}b_{i,j}))^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}) \quad (\text{e 13.38})$$

$$= \exp(-\sqrt{-1}b_{i,j}) \omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}). \quad (\text{e 13.39})$$

Note that, by (e 13.14) and (e 13.10), for  $t \in [t_i, t_{i+1}]$ ,

$$\|\omega_j(t_i)^* \omega_j(t) - 1\| < 2(m(A)^2)\varepsilon_1/16 + 2\gamma_2 < 1/32, \quad (\text{e 13.40})$$

$j = 1, 2, \dots, k(A)$ ,  $i = 0, 1, \dots, n-1$ . By Lemma 3.5 of [34],

$$(t_s \otimes \text{Tr}_{m(A)})(\log(\omega_{j,s}(t_i)^* \omega_{j,s}(t_{i+1}))) = 0. \quad (\text{e 13.41})$$

It follows that (by the Exel formula (see [22]), using (e 13.37), (e 13.39) and (e 13.41))

$$(t \otimes \text{Tr}_{m(A)})(\text{bott}_1(V_{i,j}, W_i^* W_{i+1})) \quad (\text{e 13.42})$$

$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(V_{i,j}^* W_i^* W_{i+1} V_{i,j} W_{i+1}^* W_i)) \quad (\text{e 13.43})$$

$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* W_{i+1,s} V_{i,j} W_{i+1}^*)) \quad (\text{e 13.44})$$

$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(W_i V_{i,j}^* W_i^* V_{i,j} V_{i+1,j}^* W_{i+1} V_{i+1,j} W_{i+1}^*)) \quad (\text{e 13.45})$$

$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)(t \otimes \text{Tr}_{m(A)})(\log(\exp(-\sqrt{-1}b_{i,j}) \omega_j(t_i)^* \omega_j(t_{i+1}) \exp(\sqrt{-1}b_{i+1,j}))) \quad (\text{e 13.46})$$

$$= \left(\frac{1}{2\pi\sqrt{-1}}\right)[(t \otimes \text{Tr}_{m(A)})(-\sqrt{-1}b_{i,j}) + (t \otimes \text{Tr}_{m(A)})(\log(\omega_j(t_i)^* \omega_j(t_{i+1}))) \quad (\text{e 13.47})$$

$$+ (t \otimes \text{Tr}_{m(A)})(\sqrt{-1}b_{i,j})] \quad (\text{e 13.48})$$

$$= \frac{1}{2\pi}(t \otimes \text{Tr}_{m(A)})(-b_{i,j} + b_{i+1,j}) \quad (\text{e 13.49})$$

for all  $t \in T(F_2)$ . In other words,

$$\text{bott}_1(V_{i,j}, W_i^* W_{i+1}) = -\lambda_{i,j} + \lambda_{i+1,j} \quad (\text{e 13.50})$$

$j = 1, 2, \dots, m(A)$ ,  $i = 0, 1, \dots, n-1$ .

Define  $\beta_0 = 0$ ,  $\beta_1 = [\Lambda_{0,1}] - \alpha_1 + \alpha_0 + \beta_0$ ,

$$\beta_i = [\Lambda_{i-1,i}] - \alpha_i + \alpha_{i-1} + \beta_{i-1}, \quad i = 2, 3, \dots, n. \quad (\text{e 13.51})$$

Then

$$\begin{aligned} \beta_1([g_j]) &= \Lambda_{0,1}([g_j]) - \lambda_{1,j} + \lambda_{0,j} = 0, \\ \beta_2([g_j]) &= \Lambda_{1,2}([g_j]) - \lambda_{2,j} - \lambda_{1,j} + \beta_1([g_j]) = 0 \text{ and} \\ \beta_i([g_j]) &= \lambda_{i-1,i}([g_j]) - \lambda_{i,j} - \lambda_{i-1,j} - \beta_{i-1}([g_j]) = 0, \quad i = 3, \dots, n. \end{aligned}$$

It follows 5.2.5 of [35] that there is  $\varrho \in \text{Hom}_\Lambda(\underline{K}(A), \underline{K}(F_1 \otimes \mathbb{C}))$  such that

$$\begin{aligned} \varrho(\beta(K_1(A))) &= 0 \text{ and} \\ \varrho \times ([h_1^\sim] - [h_0^\sim])|_{\beta(\underline{K}(A))} &= \beta_n|_{\beta(\underline{K}(A))}. \end{aligned}$$

Define  $\kappa_0 = \alpha_0 + \beta_0 + \varrho \times [h_0^\sim]$ ,  $\kappa_i = \alpha_i + \beta_i + \varrho \times [h_0^\sim]$ ,  $i = 1, 2, \dots, n$ . Note that, on  $\beta(\underline{K}(A))$ ,

$$\kappa_n = \alpha_n + \beta_n + \varrho \times [h_0^\sim] = \alpha_n + \varrho \times ([h_1^\sim] - [h_0^\sim]) + \varrho \times [h_0^\sim] \quad (\text{e 13.52})$$

$$= \alpha_n + \varrho \times [h_1^\sim] = (\alpha_e + \varrho) \times [h_1^\sim], \quad (\text{e 13.53})$$

and, by (e 13.31),  $\kappa_0 = \alpha_0 + \varrho \times [h_0^\sim] = \alpha_e \times [h_0^\sim] + \varrho \times [h_0^\sim]$ . We also have, for each  $j = 1, 2, \dots, k(A)$ ,

$$\begin{aligned} \kappa_i([g_j]) &= \lambda_{i,j} + (h_0^\sim)_{*0} \circ \varrho([g_j]) = \lambda_{i,j}, \quad i = 0, 1, \dots, n \text{ and} \\ (\varrho + \alpha_e)([g_j]) &= \lambda_{e,j}. \end{aligned}$$

Applying 7.4 of [18] (using (e 13.27), (e 13.3)), there are unitaries  $z_i \in F_2$ ,  $i = 1, 2, \dots, n-1$ , and  $z_e \in F_1 \otimes \mathbb{C}$  with  $z_e = z'_e \oplus 1$  such that

$$\| [z_i, \pi_{t_i} \circ \varphi^\sim(g)] \| < \delta_u \text{ for all } g \in \mathcal{G}_u \text{ and} \quad (\text{e 13.54})$$

$$\text{Bott}(z_i, \pi_{t_i} \circ \varphi^\sim) = (\kappa_i)|_{\beta(\underline{K}(A))}, \quad 1, 2, \dots, n-1, \quad (\text{e 13.55})$$

$$\| [z_e, \pi_e \circ \varphi^\sim(g)] \| < \delta_u \text{ for all } g \in \mathcal{G}_u \text{ and} \quad (\text{e 13.56})$$

$$\text{Bott}(z_e, \pi_e \circ \varphi^\sim) = (\varrho + \alpha_e)|_{\beta(\underline{K}(A))}. \quad (\text{e 13.57})$$

Put

$$z_0 = h_0(z_e) \otimes (1_{F_2} - h_0(1_{F_1})) \quad \text{and} \quad z_n = h_1(z_e) \oplus (1_{F_2} - h_1(1_{F_1})).$$

Note that, as above,

$$\text{Bott}(z_0, \pi_0 \circ \varphi^\sim) = \kappa_0|_{\beta(\underline{K}(A))} \text{ and } \text{Bott}(z_n, \pi_0 \circ \varphi^\sim) = \kappa_n|_{\beta(\underline{K}(A))}.$$

Let

$$U_i = z_i w_i w_{i+1}^* z_{i+1}, \quad i = 0, 1, \dots, n-1. \quad (\text{e 13.58})$$

Then, by (e 13.54), (e 13.56) and (e 13.32),

$$\| [U_i, \pi_{t_i} \circ \varphi^\sim(g)] \| < 2\delta_u + 2\varepsilon_1/4 < \delta_1/2 \text{ for all } g \in \mathcal{G}_u. \quad (\text{e 13.59})$$

We also compute that (using the choice of  $\delta_1$  and (e 13.51))

$$\begin{aligned}
\text{Bott}(U_i, \pi_{t_i} \circ \varphi^\sim) &= \text{Bott}(z_i, \pi_{t_i} \circ \varphi^\sim) + \text{Bott}(w_i^* w_{i+1}, \pi_{t_i} \circ \varphi^\sim) \\
&= \text{Bott}(z_{i+1}, \pi_{t_i} \circ \varphi^\sim) = \kappa_i + [\Lambda_{i,i+1}] - \kappa_{i+1} \\
&= \alpha_i + \beta_i + \varrho \times [h_0] + [\Lambda_{i,i+1}] - (\alpha_{i+1} + \beta_{i+1} + \varrho \times [h_0]) \\
&= \alpha_i + \beta_i + [\Lambda_{i,i+1}] - \alpha_{i+1} - ([\Lambda_{i,i+1}] - \alpha_{i+1} + \alpha_i + \beta_i) = 0,
\end{aligned}$$

$i = 0, 1, \dots, n-1$ . Note that, by the assumption (e 13.3),

$$t_s \circ \pi_t \circ \varphi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}'_1, \quad (\text{e 13.60})$$

where  $t_s$  is the normalized trace on  $M_{n_s}$ ,  $1 \leq s \leq F(2)$ . Then, by this, (e 13.59), (e 13.60) and by applying 6.7 of [18] we obtain a continuous path of unitaries  $\{U_i(t) : t \in [t_i, t_{i+1}]\} \subset F_2$  such that  $U_i(t_i) = 1_{F_2}$  and  $U(t_{i+1}) = z_i(w_i)^* w_{i+1} z_{i+1}^*$  and

$$\|[U_i(t), \pi_t \circ \varphi^\sim(f)]\| < \varepsilon/32 \text{ for all } f \in \mathcal{F}, \quad (\text{e 13.61})$$

$i = 0, 1, \dots, n-1$ . Now define  $W(t) = w_i z_i^* U_i(t)$  for  $t \in [t_i, t_{i+1}]$ ,  $i = 0, 1, \dots, n-1$ . Then  $W(t) \in C([0, 1], F_2)$  but also

$$W(0) = w_0 z_0^* = h_0^\sim(w_e z_e^*) \text{ and } W(1) = w_n z_n^* = h_1^\sim(w_e z_e^*).$$

Therefore  $W \in \tilde{C}$ . One then checks that, by (e 13.10), (e 13.61), (e 13.54) and (e 13.11),

$$\|W(t)(\pi_t \circ \varphi^\sim)(f)W(t)^* - (\pi_t \circ \psi^\sim)(f) \otimes 1_{M_N}\| \quad (\text{e 13.62})$$

$$< \|W(t)(\pi_t \circ \varphi^\sim)(f)W(t)^* - W(t)(\pi_{t_i} \circ \varphi^\sim)(f)W^*(t)\| \quad (\text{e 13.63})$$

$$+ \|W(t)(\pi_{t_i} \circ \varphi^\sim)(f)W(t)^* - W(t_i)(\pi_{t_i} \circ \varphi^\sim)(f)W(t_i)^*\| \quad (\text{e 13.64})$$

$$+ \|W(t_i)(\pi_{t_i} \circ \varphi^\sim)(f)W(t_i)^* - (w_i \pi_{t_i} \circ \varphi^\sim)(f)w_i^*\| \quad (\text{e 13.65})$$

$$+ \|w_i(\pi_{t_i} \circ \varphi^\sim)(f)w_i^* - \pi_{t_i} \circ \psi^\sim(f)\| \quad (\text{e 13.66})$$

$$+ \|\pi_{t_i} \circ \psi^\sim(f) - \pi_t \circ \varphi^\sim(f)\| \quad (\text{e 13.67})$$

$$< \varepsilon_1/16 + \varepsilon/32 + \delta_u + \varepsilon_1/16 + \varepsilon_1/16 < \varepsilon \quad (\text{e 13.68})$$

for all  $f \in \mathcal{F}$  and for  $t \in [t_i, t_{i+1}]$ . □

**Definition 13.3.** Let  $D$  be a non-unital  $C^*$ -algebra. Denote by  $C(\mathbb{T}, \tilde{D})^\circ$  the  $C^*$ -subalgebra of  $C(\mathbb{T}, \tilde{D})$  generated by  $C_0(\mathbb{T} \setminus \{1\}) \otimes 1_{\tilde{D}}$  and  $1_{C(\mathbb{T})} \otimes D$ . The unitization of  $C(\mathbb{T}, D)^\circ$  is  $C(\mathbb{T}, \tilde{D}) = C(\mathbb{T}) \otimes \tilde{D}$ . Let  $C$  be another non-unital  $C^*$ -algebra,  $L : C(\mathbb{T}, \tilde{D})^\circ \rightarrow C$  be a completely positive contractive linear map and  $L^\sim : C(\mathbb{T}) \otimes \tilde{D} \rightarrow \tilde{C}$  be the unitization. Denote by  $z$  the standard unitary generator of  $C(\mathbb{T})$ . For any finite subset  $\mathcal{F} \subset C(\mathbb{T}) \otimes D$ , any finite subset  $\mathcal{F}_d \subset \tilde{D}$ , and  $\varepsilon > 0$ , there exists a finite subset  $\mathcal{G} \subset D$  and  $\delta > 0$  such that, whenever  $\varphi : D \rightarrow C$  is a  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map (for any  $C^*$ -algebra  $C$ ) and  $\|[u, \varphi(g)]\| < \delta$  for all  $g \in \mathcal{G}$ , there exists a  $\mathcal{F}$ - $\varepsilon$ -multiplicative completely positive contractive linear map  $L : C(\mathbb{T}) \otimes \tilde{D} \rightarrow \tilde{D}$  such that

$$\|L(z \otimes 1) - u\| < \varepsilon \text{ and } \|L(1 \otimes d) - \varphi^\sim(d)\| < \varepsilon \text{ for all } d \in \mathcal{F}_d. \quad (\text{e 13.69})$$

We will denote such  $L$  by  $\Phi_{u, \varphi}$ .

Conversely, there exists a finite subset  $\mathcal{G}' \subset C(\mathbb{T}, \tilde{D})^\circ$  and  $\delta' > 0$ , if  $L : C(\mathbb{T}, D)^\circ \rightarrow C$  is  $\mathcal{G}'$ - $\delta'$ -multiplicative completely positive contractive linear map, there is a unitary  $u \in \tilde{C}$  such that

$$\|\tilde{L}(z \otimes 1) - u\| < \varepsilon \quad (\text{e 13.70})$$

and  $\varphi = L^\sim|_{1 \otimes D}$  is a completely positive contractive linear map.

**Lemma 13.4.** Let  $A = C(\mathbb{T}) \otimes \tilde{D}$ , where  $D \in \mathcal{B}_T$ . Let  $\mathcal{F} \subset A$  be a finite subset, let  $\varepsilon > 0$  be a positive number and let  $\Delta : A_+^{q,1} \setminus \{0\} \rightarrow (0, 1)$  be an order preserving map. There exists a finite subset  $\mathcal{H}_1 \subset A_+^{q,1} \setminus \{0\}$ ,  $\gamma_1 > 0$ ,  $\gamma_2 > 0$ ,  $\delta > 0$ , a finite subset  $\mathcal{G} \subset A$ , and a finite subset  $\mathcal{P} \subset \underline{K}(A)$ , a finite subset  $\mathcal{H}_2 \subset A$  a finite subset  $\mathcal{U} \subset J_c(K_1(A))$  for which  $[\mathcal{U}] \subset \mathcal{P}$ , satisfying the following: For any unital  $\mathcal{G}$ - $\delta$ -multiplicative contractive completely positive linear maps  $\Phi_{u,\varphi}, \Phi_{v,\psi} : A \rightarrow \tilde{C}$  for some amenable  $C \in \mathcal{D}_0$  with continuous scale, where  $u, v \in U(\tilde{C})$  and  $\varphi, \psi : D \rightarrow C$  are two  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps  $(\{g \otimes 1 : g \in \mathcal{G}_d\} \subset \mathcal{G})$  such that

$$[\Phi_{u,\varphi}]|_{\mathcal{P}} = [\Phi_{v,\psi}]|_{\mathcal{P}}, \quad (\text{e 13.71})$$

$$\tau(\Phi_{u,\varphi}(a)) \geq \Delta(\hat{a}), \quad \tau(\Phi_{v,\psi}(a)) \geq \Delta(\hat{a}) \text{ for all } \tau \in T(C) \text{ and } a \in \mathcal{H}_1, \quad (\text{e 13.72})$$

$$|\tau \circ \Phi_{u,\varphi}(a) - \tau \circ \Phi_{v,\psi}(a)| < \gamma_1 \text{ for all } a \in \mathcal{H}_2 \text{ and} \quad (\text{e 13.73})$$

$$\text{dist}(\Phi_{u,\varphi}^\dagger(y), \Phi_{v,\psi}^\dagger(y)) < \gamma_2 \text{ for all } y \in \mathcal{U}, \quad (\text{e 13.74})$$

there exists a unitary  $W \in \tilde{C}$  such that

$$\|W(\Phi_{u,\varphi}(f))W^* - (\Psi_{v,\psi}(f))\| < \varepsilon, \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.75})$$

*Proof.* Let us first reduce to the case that  $D \in \overline{\mathcal{D}_s}$ . Fix any finite subset  $\mathcal{F}_d \subset D$  and any  $\varepsilon_d > 0$ , by 6.3, there is  $D_n \in \overline{\mathcal{D}_2}$  such that

$$\text{dist}(x, D_n) < \varepsilon_d \text{ for all } x \in \mathcal{F}_d. \quad (\text{e 13.76})$$

This effectively allows us to assume that  $D \in \overline{\mathcal{D}_2}$  which has stable rank no more than 3 and  $K_i(D)$  is finitely generated. It should then be noted that  $C(\mathbb{T}, \tilde{D}) \in \overline{\mathcal{D}_3}$ .

Now we assume that  $D \in \overline{\mathcal{D}_2}$ .

Let  $\mathbf{L} = 7\pi$ ,  $r_0 = 0$ ,  $r_1 = 0$ ,  $\mathbf{T}(n, k) = n$  for all  $(n, k)$ ,  $s = 1$  and  $R = 6$ . Let  $1/2 > \varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset. Let  $\Delta_0 = \Delta/2$ . Let  $F' : A_+ \setminus \{0\} \rightarrow \mathbb{R} \times \mathbb{N}$  be given by 13.1 associated with  $\Delta_0$ .

Put  $A_0 = C(\mathbb{T}, \tilde{D})^\circ$ . Let  $\mathcal{F}_I \subset A_0$  such that, if  $x \in \mathcal{F}$ , then  $x = \lambda + y$  for some  $y \in \mathcal{F}_I$ .

Let  $\delta_0 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_0 \subset A_0$  (in place of  $\mathcal{G}$ ) be finite subset,  $\mathcal{P}_0 \subset \underline{K}(A_0)$  (in place of  $\mathcal{P}$ ),  $\mathcal{U}_0 \subset U(M_N(A))$  (for some integer  $N \geq 1$ )  $\mathcal{H}_0 \subset (A_0)_+ \setminus \{0\}$  (in place of  $\mathcal{H}$ ) and  $K \geq 1$  be an integer required by Theorem 7.9 of [17] for  $A_0$ ,  $\varepsilon/16$  (in place of  $\varepsilon$ ),  $\mathcal{F}_I$  (in place of  $\mathcal{F}$ ),  $\mathcal{L}$ ,  $F'$ , (in place  $F$ ), as well as  $r_0, r_1, T$ ,  $s$  and  $R$  above. As in the remark 7.10 of [17], we can choose  $\mathcal{U}_0 = \{g_1, g_2, \dots, g_{k(A)}\}$  so that  $K_1(A) \cap \mathcal{P}_0 = \{[g_1], [g_2], \dots, [g_{k(A)}]\}$ .

Let  $\gamma'_1 > 0$ ,  $\gamma'_2 > 0$ ,  $\delta' > 0$ ,  $\mathcal{G}' \subset A$ ,  $\mathcal{H}'_1 \subset (A_+^1) \setminus \{0\}$ ,  $\mathcal{P}' \subset \underline{K}(A)$ ,  $\overline{\mathcal{U}'} \subset J_c(K_1(A))$  and  $\mathcal{H}'_2 \subset A_{s.a.}$  be finite subsets required by 13.2 for  $\min\{\delta_0/4, \varepsilon/16\}$  (in place of  $\varepsilon$ )  $\mathcal{G}_0$  (in place of  $\mathcal{F}$ ) and for  $\Delta_0$  (in place of  $\Delta$ ).

Put  $\gamma_1 = \gamma'_1/4$ ,  $\gamma_2 = \min\{\gamma'_2/16, \varepsilon/64\}$ ,  $\delta = \min\{\delta'/16, \delta_0/16, \gamma_1/16, \gamma_2/16, \varepsilon/2^{10}\}$ ,  $\mathcal{H}_1 = \mathcal{H}'_1$ ,  $\mathcal{H}_2 = \mathcal{H}'_2$  and  $\mathcal{G} = \mathcal{G}'$ .

Now suppose that  $\Phi_1, \Phi_2 : A \rightarrow \tilde{C}$  are two  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps.

Since  $C \in \mathcal{D}_0$ , there exists a sequence of positive elements  $\{b_n\}$  of  $C$ , a sequence of  $C^*$ -algebras  $C_{0,n} \in \mathcal{R}$  (see 7.8), two sequences of completely positive contractive linear maps  $\varphi_{0,n} : A \rightarrow B_n$  and  $\varphi_{1,n} : C \rightarrow C_{0,n}$  such that

$$\lim_{n \rightarrow \infty} \|\varphi_{i,n}(ab) - \varphi_{i,n}(a)\varphi_{i,n}(b)\| = 0 \text{ for all } a, b \in C, \quad (\text{e 13.77})$$

$$\lim_{n \rightarrow \infty} \|x - \text{diag}(\varphi_{0,n}(x), \overbrace{\text{diag}(\varphi_{1,n}(x), \varphi_{1,n}(x), \dots, \varphi_{1,n}(x))}^K)\| = 0 \text{ for all } x \in C \quad (\text{e 13.78})$$

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(C)} d_\tau(b_n) = 0, \quad t(f_{1/4}(\varphi_{1,n}(e_C))) \geq 1/2 \text{ for all } t \in T(C_{0,n}) \quad (\text{e 13.79})$$

$$\text{and } \tau(f_{1/4}(\varphi_{1,n}(e_C))) > 1/2 \text{ for all } \tau \in T(C), \quad (\text{e 13.80})$$

where  $e_C \in C$  is a strictly positive element with  $\|e_C\| = 1$ ,  $B_n = \overline{b_n C b_n}$ . Put  $C_n = M_K(C_{0,n})$ ,  $n = 1, 2, \dots$ . It should be noted that  $C_n \perp B_n$ ,  $n = 1, 2, \dots$ . We may assume, without loss of generality, for all  $n$ ,

$$\sup_{\tau \in T(C)} d_\tau(b_n) < \min\{\gamma_1/64K, \gamma_2/64K, \min\{\Delta_0(\hat{h}) : h \in \mathcal{H}_1\}/4(K+2)\}. \quad (\text{e 13.81})$$

Let  $u_i, v_i \in M_N(\tilde{C})$  ( $i = 1, 2, \dots, k(A)$ ) be two unitaries such that

$$\|(\Phi_1 \otimes \text{id}_{M_N})(g_i) - u_i\| < \min\{\varepsilon/2^8, \gamma_2/8\} \text{ and } \|(\Phi_2 \otimes \text{id}_{M_N})(g_i) - v_i\| < \min\{\varepsilon/2^8, \gamma_2/8\}.$$

Let  $W_i \in CU(\tilde{C})$  be such that

$$\|u_i v_i^* - w_i\| < (5/4)\gamma_2 \text{ and } w_i = \prod_{j=1}^{m(i)} w_{i,j}, \quad w_{i,j} = w_{1,i,j}^* w_{2,i,j}^* w_{1,i,j} w_{2,i,j}, \quad (\text{e 13.82})$$

where  $w_{s,i,j} \in U(\tilde{C})$ ,  $s = 1, 2$ ,  $j = 1, 2, \dots, m(i)$  and  $i = 1, 2, \dots, k(A)$ . Let  $m = \max\{m(i) : 1 \leq i \leq k(A)\}$ .

Write  $w_{s,i,j} = \alpha_{s,i,j} + c(w_{s,i,j})$ , where  $\alpha_{s,i,j} \in \mathbb{T} \subset \mathbb{C}$  and  $c(w_{s,i,j}) \in C$ ,  $j = 1, 2, \dots, m(i)$ . Note that  $\|c(w_{s,i,j})\| \leq 2$ ,  $j = 1, 2, \dots, m(i)$ ,  $i = 1, 2, \dots, k(A)$ .

Define  $\psi_{1,n} : A \rightarrow C_n$  by  $\psi_{1,n}(a) = \text{diag}(\overbrace{\varphi_{1,n}(a), \varphi_{1,n}(a), \dots, \varphi_{1,n}(a)}^K)$  for all  $n$ . Put  $\Psi_j = \psi_{1,n} \circ \Phi_j : A \rightarrow C_n$ ,  $j = 1, 2$ .

Let  $\mathcal{G}_2 = \mathcal{G} \cup \{c(w_{s,i,j}) : s = 1, 2, 1 \leq j \leq m(i), 1 \leq i \leq k(A)\}$ . We can choose  $n$  large enough so that  $\psi_{0,n}$  and  $\psi_{1,n}$  are  $\mathcal{G}_2$ - $\delta/2^{12}mN^2$ -multiplicative. In particular,

$$\text{dist}(\overline{\varphi_{0,n}^\sim(u_i)}, \overline{\varphi_{0,n}^\sim(v_i^*)}) \leq \gamma'_2/4 \text{ in } U(\tilde{B}_n)/CU(\tilde{B}_n) \text{ and} \quad (\text{e 13.83})$$

$$\text{dist}(\overline{\psi_{1,n}^\sim(u_i)}, \overline{\psi_{1,n}^\sim(v_i^*)}) \leq \gamma'_2/4 \text{ in } U(\tilde{C}_n)/CU(\tilde{C}_n). \quad (\text{e 13.84})$$

It is standard to check that, by choosing sufficiently large  $n$ , we may assume that  $\Psi_j$  are  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear maps satisfying the following:

$$t \circ \Psi_1(h) \geq \Delta_0(\hat{h}), \quad t \circ \Psi_2(h) \geq \Delta_0(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \quad (\text{e 13.85})$$

$$|t \circ \Psi_1(g) - t \circ \Psi_2(g)| < \gamma'_2 \text{ for all } g \in \mathcal{H}_2. \quad (\text{e 13.86})$$

Combining these with (e 13.84), by applying 13.2, one obtains a unitary  $U_1 \in \tilde{C}_n$  such that

$$\|U_1^* \Psi_1(x) U_1 - \Psi_2(x)\| < \min\{\delta_0/4, \varepsilon/4\} \text{ for all } x \in \mathcal{G}_0. \quad (\text{e 13.87})$$

Write  $U_1 = \lambda \cdot 1_{\tilde{C}_n} + c(U_1)$ , where  $\lambda \in \mathbb{T} \subset \mathbb{C}$  and  $c(U_1) \in C_n$ . Define  $V_1 = \lambda \cdot 1_{\tilde{C}} + c(U_1)$ . Then  $V_1 \in U(\tilde{C})$ . Note, since  $B_n \perp C_n$ ,  $V_1^* b V_1 = b$  for all  $b \in B_n$ .

Let  $E_n = \overline{C_{0,n} C C_{0,n}}$  and  $e_{E_n}$  be a strictly positive element with  $\|e_{E_n}\| = 1$ . Put  $\Lambda : A \rightarrow C_{1,n} \subset E_n$  by defining  $\Lambda(a) = \text{Ad } V_1 \circ \varphi_{1,n} \circ \Phi_1(a)$  for all  $a \in A_0$ . By (e 13.85),  $\Lambda$  is  $F'$ - $\mathcal{H}_1$ -full in  $C_{1,n}$ . It follows it is  $F'$ - $\mathcal{H}_1$ -full in  $E_n$ . By (e 13.81), we may assume that  $b_n \lesssim e_{E_n}$ .

Let  $L_i = \varphi_{0,n} \circ \Phi_i$ ,  $i = 1, 2$ . By (e 13.78), we also assume that  $L_i$  is also  $\mathcal{G}$ - $2\delta$ -multiplicative and

$$\|L_i(x) \oplus \Psi_i(x) - \Phi_i(x)\| < \delta \text{ for all } x \in \mathcal{G}. \quad (\text{e 13.88})$$

Since  $K_i(C_n) = \{0\}$ ,  $i = 0, 1$ , we conclude that

$$[L_1]|_{\mathcal{P}} = [L_2]|_{\mathcal{P}}. \quad (\text{e 13.89})$$

It follows from 4.4 and (e 13.83) that, in  $B_n$ ,

$$\text{cel}(\lceil L_1(z \otimes 1) \rceil \lceil L_2(z \otimes 1) \rceil^*) < 7\pi = \mathbf{L}. \quad (\text{e 13.90})$$

It follows from 7.9 of [17] that there exists a unitary  $W_1 \in \tilde{B}$  such that

$$\|W_1^* \text{diag}(L_1(a), S(a))W_1 - \text{diag}(L_2, S(a))\| < \varepsilon/16, \quad (\text{e 13.91})$$

where  $S(a) = \text{diag}(\overbrace{\Lambda(a), \Lambda(a), \dots, \Lambda(a)}^K) = V_1^* \Psi_1(a) V_1$ , for all  $a \in \mathcal{F}_I$ . Put  $W = V_1 W_1$ . One then estimates, by (e 13.88), (e 13.91) and (e 13.87),

$$\text{Ad } W \circ \Phi_1 \approx_\delta \text{Ad } W \circ (L_1 \oplus \text{Ad } V_1 \circ \Psi_1) \quad (\text{e 13.92})$$

$$\approx_{\varepsilon/16} L_2 \oplus V_1 \circ \Psi_1 \approx_{\varepsilon/4} L_2 \oplus \Psi_2 \approx_\delta \Phi_2 \text{ on } \mathcal{F}_I. \quad (\text{e 13.93})$$

Therefore

$$\|\text{Ad } W \circ \Phi_1(a) - \Phi_2(a)\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad (\text{e 13.94})$$

□

**Lemma 13.5.** *Let  $A \in \mathcal{B}_T$  have continuous scale. For any  $1 > \varepsilon > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exist  $\delta > 0$ ,  $\sigma > 0$ , a finite subset  $\mathcal{G} \subset A$ , a finite subset  $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$  of projections of  $M_N(\tilde{A})$  (for some integer  $N \geq 1$ ) such that  $\{[p_1] - [q_1], [p_2] - [q_2], \dots, [p_k] - [q_k]\}$  generates a free subgroup  $G_u$  of  $K_0(A)$ , and a finite subset  $\mathcal{P} \subset \underline{K}(A)$ , satisfying the following:*

*Suppose that  $\varphi : A \rightarrow B \otimes U$  is a homomorphism which maps strictly positive elements to strictly positive elements, where  $B \in \mathcal{D}_0$  has continuous scale and  $U$  is a UHF-algebra of infinite type. If  $u \in U(\widetilde{B \otimes U})$  is a unitary such that*

$$\|[\varphi(x), u]\| < \delta \text{ for all } x \in \mathcal{G}, \quad (\text{e 13.95})$$

$$\text{Bott}(\varphi, u)|_{\mathcal{P}} = 0, \quad (\text{e 13.96})$$

$$\text{dist}(\langle ((1 - \varphi(p_i)) + \varphi(p_i)u)(1 - \varphi(q_i)) + \varphi(q_i)u^* \rangle, \bar{1}) < \sigma \text{ and} \quad (\text{e 13.97})$$

$$\text{dist}(\bar{u}, \bar{1}) < \sigma, \quad (\text{e 13.98})$$

*then there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1]\} \subset U_0(\widetilde{B \otimes U})$  such that*

$$u(0) = u, \quad u(1) = 1 \quad (\text{e 13.99})$$

$$\|[\varphi(a), u(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F} \text{ and for all } t \in [0, 1]. \quad (\text{e 13.100})$$

*Proof.* Without loss of generality, one only has to prove the statement with assumption that  $u \in CU(B \otimes U)$ . Since  $B \otimes U \otimes U \cong B \otimes U$ , to simplify notation, without loss of generality, we may assume that  $B = B \otimes U$ . In particular,  $K_0(\tilde{B})$  is weakly unperforated (see 5.5).

In what follows we will use the fact that every C\*-algebra in  $\mathcal{D}_0$  has stable rank one. Let

$$\Delta_0(h) = \inf\{\tau(h) : \tau \in T(A)\} \text{ for all } h \in A_+^1 \setminus \{0\} \text{ and} \quad (\text{e 13.101})$$

$$\Delta_T(f) = (1/4) \int f dm \text{ for all } f \in C(\mathbb{T})_+^1 \setminus \{0\}, \quad (\text{e 13.102})$$

where  $m$  is the normalized Lebesgue measure on the unit circle  $\mathbb{T}$ . Let  $A_2 = C(\mathbb{T}) \otimes \tilde{A}$ . Define

$$\Delta(\hat{h}) = \sup\{\Delta_0(h_1)\Delta_T(h_2) : h \geq h_1 \otimes h_2 : h_1 \in A_+^1 \setminus \{0\} \text{ and } h_2 \in C(\mathbb{T})_+^1 \setminus \{0\}\} \quad (\text{e 13.103})$$

for  $h \in (A_2)_+^1 \setminus \{0\}$ . Let  $\mathcal{F}_1 = \{x \otimes f : x \in \mathcal{F}, f = 1, z, z^*\}$ . To simplify notation, without loss of generality, we may assume that  $\mathcal{F} \subset A^1$ . Let  $1 > \delta_1 > 0$  (in place of  $\delta$ ),  $\mathcal{G}_1 \subset A_2$  be a finite



subset (in place of  $\mathcal{G}$ ),  $1/4 > \gamma_1 > 0$ ,  $1/4 > \gamma_2 > 0$ ,  $\mathcal{P}' \subset \underline{K}(A_2)$  (in place of  $\mathcal{P}$ ) be a finite subset,  $\mathcal{H}_1 \subset (A_2)_+^1 \setminus \{0\}$  be a finite subset,  $\mathcal{H}_2 \subset (A_2)_{s.a.}$  be a finite subset and  $\mathcal{U} \subset J_c(K_1(A_2))$  (for some integer  $N \geq 1$ ) be a finite subset as required by 13.4 for  $\varepsilon/16$  (in place of  $\varepsilon$ ),  $\mathcal{F}_1$  (in place of  $\mathcal{F}$ ),  $\Delta$  and  $A_2$  (in place of  $A$ ). Here we assume that  $[L]|_{\mathcal{P}'}$  is well defined whenever  $L$  is a  $\mathcal{G}_1$ - $\delta_1$ -multiplicative completely positive contractive linear map from  $A_2$ . Moreover,

$$[L_1]|_{\mathcal{P}'} = [L_2]|_{\mathcal{P}'}, \quad (\text{e 13.104})$$

if both  $L_1$  and  $L_2$  are  $\mathcal{G}_1$ - $\delta_1$ -multiplicative completely positive contractive linear maps from  $A_2$  to a unital  $C^*$ -algebra and  $\|L_1(g) - L_2(g)\| < \delta_1$  for all  $g \in \mathcal{G}_1$ .

Without loss of generality, we may assume that  $\mathcal{G}_1 = \{z \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes a : a \in \mathcal{G}_{1A}\}$ ,  $\mathcal{H}_1 = \{h' \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes h'' : h' \in \mathcal{H}_{1T} \text{ and } h'' \in \mathcal{H}_{1A}\}$ ,  $\mathcal{H}_2 = \{h_1 \otimes 1_{\tilde{A}}, 1_{C(\mathbb{T})} \otimes h_2 : h_1 \in \mathcal{H}_{2T} \text{ and } h_2 \in \mathcal{H}_{2A}\}$ , where  $\mathcal{H}_{1T} \subset C(\mathbb{T})_+^1 \setminus \{0\}$ ,  $\mathcal{H}_{2T} \subset C(\mathbb{T})_{s.a.}$ ,  $\mathcal{G}_{1,A} \subset \tilde{A}$ ,  $\mathcal{H}_{1A} \subset A_+^1 \setminus \{0\}$  and  $\mathcal{H}_{2A}$  are finite subsets. Furthermore, we may also assume that elements in  $\mathcal{H}_{1T}$  and  $\mathcal{H}_{2T}$  are polynomials of  $z$  and  $z^*$  of degree no more than  $N_1$  and all coefficients with absolute values no more than  $M$ . In addition, we assume that  $\mathcal{H}_{1A} \subset \mathcal{H}_{2A}$ . We may assume that  $\mathcal{P}' = \mathcal{P}_1 \cup \beta(\mathcal{P}_2) \cup \beta([1_{\tilde{A}}])$ , where  $\mathcal{P}_1, \mathcal{P}_2 \subset \underline{K}(A)$  are finite subsets. We further assume that

$$\text{Bott}(\varphi, v(0))|_{\mathcal{P}_2} = \text{Bott}(\varphi, v(t))|_{\mathcal{P}_2}, \quad (\text{e 13.105})$$

if  $\|[\varphi(a), v(t)]\| < \delta_1$  for all  $a \in \mathcal{G}_{1A}$  and for any continuous path of unitaries  $\{v(t) : t \in [0, 1]\}$ .

We may further assume that,

$$\mathcal{U} = \mathcal{U}_1 \cup \{\overline{1 \otimes z}\} \cup \mathcal{U}_2, \quad (\text{e 13.106})$$

where  $\mathcal{U}_1 = \{\overline{1_{C(\mathbb{T})} \otimes a} : a \in \mathcal{U}'_1 \subset U(\tilde{A})\}$  and  $\mathcal{U}'_1$  is a finite subset,  $\mathcal{U}_2 \subset U(M_N(A_2))/CU(M_N(A_2))$  is a finite subset whose elements represent a finite subset of  $\beta(K_0(A))$ . So we may assume that  $\mathcal{U}_2 \in J_c(\beta(K_0(A)))$ .

We may assume that  $\mathcal{U}_2 = \mathcal{U}_{2f} \sqcup \mathcal{U}_{2t}$ , where  $\mathcal{U}_{2f} = \{J_c(g_{1,f}), J_c(g_{2,f}), \dots, J_c(g_{m(f),f})\}$  and  $\mathcal{U}_{2t} = \{J_c(g_{1,t}), J_c(g_{2,t}), \dots, J_c(g_{m(t),t})\}$ , where  $\mathcal{P}' \cap \beta(K_0(A)) = \{g_{i,f}, g_{j,t} : 1 \leq i \leq m(f), 1 \leq j \leq m(t)\}$ . Moreover,  $\{g_{1,f}, g_{2,f}, \dots, g_{m(f),f}\}$  is a set of free generators of a finitely generated free subgroup of  $\beta(K_0(A))$  and  $\{g_{1,t}, g_{2,t}, \dots, g_{m(t),t}\}$  are generators for a finite subgroup of  $\beta(K_0(A))$ . Since  $J_c$  is a homomorphism, we may assume that there is an integer  $k_m \geq 1$  such that  $k_m J_c(g_{j,t}) = 0$  in  $U(M_N(A_2))/CU(M_N(A_2))$ . Without loss of generality, we may write that

$$g_{i,f} = [(1 \otimes (1 - p_i) + z \otimes p_i))(1 \otimes (1 - q_i) + z^* \otimes q_i)], \quad i = 1, 2, \dots, m(f). \quad (\text{e 13.107})$$

Write  $p_s = (a_{i,j}^{p_s})_{N \times N}$  and  $q_s = (a_{i,j}^{q_s})_{N \times N}$  as matrices over  $\tilde{A}$ . Let  $w_l = (b_{i,j}^l)_{N \times N}$  be unitaries in  $M_N(\tilde{A})$  such that  $\overline{w_l} = J_c(g_{j,t})$ ,  $l = 1, 2, \dots, m(t)$ .

We assume that  $(2\delta_1, \mathcal{P}, \mathcal{G}_1)$  is a  $KL$ -triple for  $A_2$ ,  $(2\delta_1, \mathcal{P}_1, \mathcal{G}_{1A})$  is a  $KL$ -triple for  $A$  (see 2.12 of [17], for example). We may also choose  $\sigma_1$  and  $\sigma_2$  such that

$$0 < \sigma_1 < (1/4) \min\{\gamma_1/16, \inf\{\Delta(\hat{f}) : f \in \mathcal{H}_1\}\}/4M(N+1) \text{ and} \quad (\text{e 13.108})$$

$$\sigma_2 = 1 - \gamma_2/16(N+1)M. \quad (\text{e 13.109})$$

. Choose  $\delta_2 > 0$  and a finite subset  $\mathcal{G}_{2A} \subset \tilde{A}$  (and denote  $\mathcal{G}_2 := \{g \otimes f : g \in \mathcal{G}_{2A}, f = \{1, z, z^*\}\}$ ) such that, for any two unital  $\mathcal{G}_2$ - $\delta_2$ -multiplicative contractive completely positive linear maps  $\Psi_1, \Psi_2 : C(\mathbb{T}) \otimes \tilde{A} \rightarrow \tilde{C}$  (any unital  $C^*$ -algebra  $C$ ), any  $\mathcal{G}_{2A}$ - $\delta_2$ -multiplicative contractive completely positive linear map  $\Psi_0 : \tilde{A} \rightarrow \tilde{C}$  and unitary  $V \in \tilde{C}$  ( $1 \leq i \leq k$ ), if

$$\|\Psi_0(g) - \Psi_1(g \otimes 1)\| < \delta_2 \text{ for all } g \in \mathcal{G}_{2A} \quad (\text{e 13.110})$$

$$\|\Psi_1(z \otimes 1_{\tilde{A}}) - V\| < \delta_2 \text{ and } \|\Psi_1(g) - \Psi_2(g)\| < \delta_2 \text{ for all } g \in \mathcal{G}_2, \quad (\text{e 13.111})$$

then

$$[(1 - \Psi_0(p_i) + \Psi_0(p_i)V)(1 - \Psi_0(q_i) + \Psi_0(q_i)V^*)] \quad (\text{e 13.112})$$

$$\approx \frac{\gamma_2}{2^{10}} [\Psi_1(((1 - p_i) + z \otimes p_i)((1 - q_i) + z^* \otimes q_i))], \quad (\text{e 13.113})$$

$$\|[\Psi_1(x)] - [\Psi_2(x)]\| < \gamma_2/2^{10} \text{ for all } x \in \mathcal{U}'_2, \quad (\text{e 13.114})$$

$$\Psi_1(((1 - p_i) + z \otimes p_i)(1 - q_i) + z^* \otimes q_i)) \quad (\text{e 13.115})$$

$$\approx \frac{\gamma_2}{2^{10}} \Psi_1(((1 - p_i) + z \otimes p_i))\Psi_1((1 - q_i) + z^* \otimes q_i)), \quad (\text{e 13.116})$$

furthermore for  $d_i^{(1)} = p_i$ ,  $d_i^{(2)} = q_i$ , there are projections  $\bar{d}_i^{(j)} \in M_N(\tilde{C})$  and unitaries  $\bar{z}_i^{(j)} \in \bar{d}_i^{(j)} M_N(\tilde{C}) \bar{d}_i^{(j)}$  such that

$$\Psi_1(((1 - d_i^{(j)}) + z \otimes d_i^{(j)})) \approx \frac{\gamma_2}{2^{12}} (1 - \bar{d}_i^{(j)}) + \bar{z}_i^{(j)} \text{ and} \quad (\text{e 13.117})$$

$$\bar{d}_i^{(j)} \approx \frac{\gamma_2}{2^{12}} \Psi_1(d_i^{(j)}), \bar{z}_i^{(1)} \approx \frac{\gamma_2}{2^{12}} \Psi_1(z \otimes p_i), \text{ and } \bar{z}_i^{(2)} \approx \frac{\gamma_2}{2^{12}} \Psi_1(q_i \otimes z^*), \quad (\text{e 13.118})$$

where  $1 \leq i \leq k$ ,  $j = 1, 2$ .

Let  $\delta_3 > 0$  and let  $\mathcal{G}_3 \subset C(\mathbb{T}, \tilde{A})^\circ$  be a finite subset required by 10.6 for  $C = C(\mathbb{T}, \tilde{A})^\circ$ ,  $\gamma_2/2$  (in place of  $\varepsilon$ ) and all unitaries in  $\mathcal{U}_{2t}$ . Without loss of generality, we may write  $\mathcal{G}_3 = \mathcal{G}_{3A} \cup \{1, z, z^*\}$ , where  $\mathcal{G}_{3A}$  is a finite subset of  $A$ .

Choose  $\delta_A = \min\{\varepsilon/16, \delta_1/16, \delta_2/16, \sigma_1/4, \sigma_2/4\}/8M(N+1)^3$  and

$$\mathcal{G}_A = \mathcal{F} \cup \mathcal{G}_{1A} \cup \mathcal{G}_{2A} \cup \mathcal{H}_{1A} \cup \mathcal{H}_{2A} \cup \mathcal{U}'_1 \cup \{a_{i,j}^{p_s}, a_{i,j}^{q_s}, b_{i,j}^l : 1 \leq s \leq 1, 1 \leq l \leq m(t), 1 \leq i, j \leq N\}.$$

Let  $\mathcal{G}'_A \subset A$  be a finite subset such that every element  $a \in \mathcal{G}_A$  has the form  $a = \lambda + b$  for some  $\lambda \in \mathbb{C}$  and  $b \in \mathcal{G}'_A$ . Let  $\mathcal{G}_4 = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \mathcal{G}_3 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{U}_1$ .

Let  $\delta_4 > 0$  (in place of  $\delta_1$ ) and a finite subset  $\mathcal{G}_5$  (in place of  $\mathcal{G}_1$ ) be as required by 2.10 of [17] for  $A$  (in place of  $C$ ),  $\delta_1/4$  (in place of  $\delta$ ),  $\delta_A$  (in place of  $\delta_c$ ),  $\sigma_1, \sigma_2, \mathcal{H}_1, \mathcal{H}_2, \mathcal{G}_4$  (in place of  $\mathcal{G}$ ),  $\mathcal{G}_A$  (in place of  $\mathcal{G}_c$ ) and  $N_1$ .

By choosing even smaller  $\delta_4$ , without loss of generality, we may assume that  $\mathcal{G}_5 = \{a \otimes f : g \in \mathcal{G}_{5A} \text{ and } f = 1, z, z^*\}$  with a large finite subset  $\mathcal{G}_{5A} \supset \mathcal{G}_A$ . Let  $\mathcal{G}'_{5A} \subset A$  be a finite subset such that every element  $g \in \mathcal{G}_{5A}$  has the form  $g = \lambda + x$  for some  $\lambda \in \mathbb{C}$  and  $x \in \mathcal{G}'_{5A}$ .

Choose  $\sigma > 0$  so it is smaller than  $\min\{\sigma_1/16, \varepsilon/16, \sigma_2/16, \delta_2/16, \delta_3/16, \delta_4/16, \delta_A/4\}$ .

Let  $\delta = \sigma$  and  $\mathcal{G} = \mathcal{G}'_{5A} \cup \mathcal{G}_A$ .

Now suppose that  $\varphi : A \rightarrow B$  is a homomorphism and  $u \in CU(\tilde{B})$  which satisfy the assumption (e 13.95) to (e 13.97) for the above mentioned  $\delta, \sigma, \mathcal{G}, \mathcal{P}, p_i$ , and  $q_i$ . There is an isomorphism  $s : U \otimes U \rightarrow U$ . Moreover,  $s \circ \iota$  is approximately unitarily equivalent to the identity map on  $U_2$ , where  $\iota : U \rightarrow U \otimes U$  defined by  $\iota(a) = a \otimes 1$  (for all  $a \in U$ ). To simplify notation, without loss of generality, we may assume that  $\varphi(A) \subset B \otimes 1 \subset B \otimes U$ . Suppose that  $u \in U(B) \otimes 1_U$  is a unitary which satisfies the assumption. As mentioned at the beginning, we may assume that  $u \in CU(B) \otimes 1_U$ .

Applying 20.10 of [17], we obtain  $e \in (B)_+$  with  $\|e\| = 1$  and  $h \in U_{s.a.}$  satisfying the conclusions of 20.10 of [17]. Note that we may assume, without loss of generality, that

$$e\varphi^\sim(g) = \varphi^\sim(g)e \text{ for all } g \in \mathcal{G}_{3A} \cup \mathcal{G}_{5A} \text{ and} \quad (\text{e 13.119})$$

$$e\varphi(g) = \varphi(g)e = \varphi(g) \text{ for all } g \in \mathcal{G}'_{3A} \cup \mathcal{G}'_{5A}. \quad (\text{e 13.120})$$

In particular, for  $E = \text{diag}(\overbrace{e, e, \dots, e}^N)$  and  $y = p_i, q_i, i = 1, 2, \dots, m(f)$ ,

$$(\varphi^\sim \otimes \text{id}_{M_N})(y)E = E(\varphi^\sim \otimes \text{id}_{M_N})(y). \quad (\text{e 13.121})$$

Put  $v_1 = u \exp(i e \otimes h)$  and  $v_2 = \exp(i e \otimes h)$ . Note that  $\text{sp}(h) = [-\pi, \pi]$  and  $t_U(h) = 0$  and where  $t_U$  is the unique tracial state of  $U$ . Let  $L_1, L_2 : C(\mathbb{T}) \otimes \tilde{A} \rightarrow \widetilde{B \otimes U}$  be given by 20.10 of [17] such that

$$|\tau(L_1(f)) - \tau(L_2(f))| < \sigma_1 \text{ for all } f \in \mathcal{H}_2, \tau \in T(B), \quad (\text{e 13.122})$$

$$\tau(g(v_1)) \geq \sigma_2 \left( \int g dm \right) \text{ for all } g \in \mathcal{H}_1, \tau \in T(B), \text{ and} \quad (\text{e 13.123})$$

$$\|L_i(c \otimes 1_{C(\mathbb{T})}) - \varphi^\sim(c) \otimes 1_U\| < \delta_A \text{ for all } c \in \mathcal{G}_c, i = 1, 2, \quad (\text{e 13.124})$$

$$\|L_1(c \otimes z^j) - \varphi^\sim(c)(u \exp(i e \otimes h))^j\| < \delta_A \text{ for all } c \in \mathcal{G}_c \text{ and} \quad (\text{e 13.125})$$

$$\|L_2(c \otimes z^j) - \varphi(c)^\sim \exp(i e \otimes h)^j\| < \delta_A \text{ for all } c \in \mathcal{G}_c \quad (\text{e 13.126})$$

and for all  $0 < |j| \leq N_1$ , where  $\varphi^\sim : \tilde{A} \rightarrow \widetilde{B \otimes U}$ . Note by (e 13.124), (e 13.125) and (e 13.126), we may write  $L_1 = \Phi_{v_1, \varphi}$  and  $L_2 = \Phi_{v_2, \varphi}$ . Let  $u(t) = \exp(i 3t(e \otimes h))$  for  $t \in [0, 1/3]$ . Then

$$\|[\varphi(a), u(t)]\| < \delta_c \text{ for all } a \in \mathcal{G}_c. \quad (\text{e 13.127})$$

In particular,

$$\text{Bott}(\varphi, v_1)|_{\mathcal{P}_2} = 0. \quad (\text{e 13.128})$$

Exactly the same reason, we have that

$$\text{Bott}(\varphi, v_2)|_{\mathcal{P}_2} = 0. \quad (\text{e 13.129})$$

This implies

$$[L_1]|_{\beta(\mathcal{P}_2)} = [L_2]|_{\beta(\mathcal{P}_2)}. \quad (\text{e 13.130})$$

We also have

$$[L_1]|_{\mathcal{P}_1} = [\varphi]|_{\mathcal{P}_1} = [L_2]|_{\mathcal{P}_1} \text{ and } [v_1] = [v_2] = 0. \quad (\text{e 13.131})$$

Therefore

$$[L_1]|_{\mathcal{P}'} = [L_2]|_{\mathcal{P}'}. \quad (\text{e 13.132})$$

Then, by (e 13.123) and the choice of  $\delta_A$ , we compute (as in (e 20.91) of [17]) that

$$\tau(L_i(h)) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, i = 1, 2. \quad (\text{e 13.133})$$

We also have

$$\text{dist}(L_1^\dagger(x), L_2^\dagger(x)) < 2\delta_A \text{ for all } x \in \mathcal{U}_1 \cup \{\overline{z \otimes 1_{\tilde{A}}}\}. \quad (\text{e 13.134})$$

Write  $V_2 = \text{diag}(\overbrace{v_2, v_2, \dots, v_2}^N)$  and  $H = \text{diag}(\overbrace{h, h, \dots, h}^N)$ . As always, we will write  $\varphi^\sim(y)$  for  $\varphi^\sim \otimes \text{id}_{M_N}(y)$  for  $y = p_i, q_i, i = 1, 2, \dots, m(f)$ . By (e 13.121),

$$\psi^\sim(p_i)V_2 = \exp(i\psi^\sim(p_i)E \otimes H) \text{ and } \psi^\sim(q_i)V_2 = \exp(i\psi^\sim(q_i)E \otimes H), \quad (\text{e 13.135})$$

$i = 1, 2, \dots, m(f)$ . However,

$$\tau(\psi(q_i)E \otimes H) = \tau(\psi(q_i)E)\tau_U(H) = 0 \text{ for all } \tau \in T(B \otimes U). \quad (\text{e 13.136})$$

In the next few lines,  $\mathbf{1} = 1_{M_N}$ . Therefore

$$\psi^\sim(p_i)V_2 + (\mathbf{1} - \psi^\sim(p_i)), \psi^\sim(q_i)V_2 + (\mathbf{1} - \psi^\sim(q_i)) \in CU(M_N(\widetilde{B \otimes U})),$$

$i = 1, 2, \dots, m(f)$ . This implies that

$$L_2^\dagger(x) = \bar{1} \text{ for all } x \in \mathcal{U}_{2f}. \quad (\text{e 13.137})$$

with  $x = ((\mathbf{1} - p_i) + p_i \otimes z)((\mathbf{1} - q_i) + q_i \otimes z^*)$ , one then computes from (e 13.116) and from the assumption (e 13.97) that

$$\overline{\langle L_1(x) \rangle} \approx_{\gamma_2/2^{10}} \overline{(\bar{z}_i^{(1)} \otimes v_2 + (\mathbf{1} - \bar{p}_i))(\bar{z}_i^{(2)} \otimes v_2 + (\mathbf{1} - \bar{q}_i))} \quad (\text{e 13.138})$$

$$= \overline{(\bar{z}_i^{(1)} + (\mathbf{1} - \bar{p}_i))(\bar{p}_i V_2 + (\mathbf{1} - \bar{p}_i) \otimes 1_U)(\bar{z}_i^{(2)} + (\mathbf{1} - \bar{q}_i))(\bar{q}_i V_2 + (\mathbf{1} - \bar{q}_i))} \quad (\text{e 13.139})$$

$$= \overline{(\bar{z}_i^{(1)} + (\mathbf{1} - \bar{p}_i))(\bar{z}_i^{(2)} + (\mathbf{1} - \bar{q}_i))} \approx_\sigma \bar{1}. \quad (\text{e 13.140})$$

where  $\bar{p}_i, \bar{q}_i, \bar{z}_i^{(1)}, \bar{z}_i^{(2)}$  are as above (see the lines below (e 13.116)), replacing  $\Psi_1$  by  $L_1$ . It follows that

$$\text{dist}(L_1^\dagger(x), \bar{1}) < \gamma_2/4 \text{ for all } x \in \{\overline{\mathbf{1} \otimes z}\} \cup \mathcal{U}_{2f}. \quad (\text{e 13.141})$$

By the choice of  $\delta_3$  and  $\mathcal{G}_4$ , and by applying 10.6, we also have

$$\text{dist}(\overline{\langle L_1(w_l) \rangle}, \overline{\langle L_2(w_l^*) \rangle}) < \gamma_2/2, \quad l = 1, 2, \dots, m(t). \quad (\text{e 13.142})$$

Combing (e 13.137), (e 13.141) and (e 13.142), we obtain that

$$\text{dist}(L_1^\dagger(w), L_2^\dagger(w)) < \gamma_2 \text{ for all } w \in \mathcal{U}. \quad (\text{e 13.143})$$

By (e 13.132), (e 13.122), (e 13.133) and (e 13.143), and by applying 13.4, we obtain a unitary  $W \in \widetilde{B \otimes U}$  such that

$$\|W^* L_2(f) W - L_1(f)\| < \varepsilon/16 \text{ for all } f \in \mathcal{F}_1. \quad (\text{e 13.144})$$

Therefore

$$\|[L_1(a), W^* v_2 W]\| < \varepsilon/8 \text{ and } \|L_1(a) - W^* L_1(a) W\| < \varepsilon/8 \text{ for all } a \in \mathcal{F} \quad (\text{e 13.145})$$

$$\text{and } \|v_1 - W^* v_2 W\| < \varepsilon/8. \quad (\text{e 13.146})$$

Let  $v_1^* W^* v_2 W = \exp(ih_1)$  for some  $h_1 \in \tilde{B}_{s.a.}$  such that  $\|h_1\| \leq 2 \arcsin(\varepsilon/16)$ . Now define  $u(t) = u \exp(i3t(e \otimes h))$  for  $t \in [0, 1/3]$ ,  $u(t) = u(1/3) \exp(i3(t - 1/3)h_1)$  for  $t \in (1/3, 2/3]$  and  $u(t) = u(2/3)W^* \exp(i3(t - 2/3)(e \otimes h)W)$  for  $t \in (2/3, 1]$ . So  $\{u(t) : t \in [0, 1]\}$  is a continuous path of unitaries in  $\widetilde{B \otimes U}$  such that  $u(0) = u$  and  $u(1) = 1_{\tilde{B}}$ . Moreover, we estimates, by (e 13.95), (e 13.145) and (e 13.145) that

$$\|[\varphi(a), u(t)]\| < \varepsilon \text{ for all } a \in \mathcal{F}. \quad (\text{e 13.147})$$

□

**Lemma 13.6.** *Let  $A \in \mathcal{B}_T$  have continuous scale. For any finite subset  $\mathcal{P} \subset \underline{K}(A)$ , there exists  $\delta_0 > 0$  and a finite subset  $\mathcal{G}_0 \subset A$  satisfy the following: For any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subset A$  and any homomorphism  $\varphi : A \rightarrow B = B_1 \otimes Q$  which maps strictly positive elements to strictly positive elements, where  $B_1 \cong B_1 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$  has continuous scale, suppose that  $u \in U(\tilde{B})$  satisfies*

$$\|[\varphi(g), u]\| < \delta_0 \text{ for all } g \in \mathcal{G}_0. \quad (\text{e 13.148})$$

*Then there exists another unitary  $v \in U(\tilde{B})$  such that*

$$\|[\varphi(g), v]\| < \min\{\varepsilon, \delta_0\} \text{ for all } g \in \mathcal{G}_0 \cup \mathcal{F} \text{ and} \quad (\text{e 13.149})$$

$$\text{Bott}(\varphi, uv)|_{\mathcal{P}} = 0 \text{ and } [uv] = 0 \text{ in } K_1(B). \quad (\text{e 13.150})$$

*Proof.* Define  $\Delta_1(\hat{h}) = \inf\{\tau(h) : \tau \in T(A)\}$  for  $h \in A_+^1 \setminus \{0\}$ . Let  $\Delta = \Delta_1/2$ . Let  $T : A_+^1 \setminus \{0\} \rightarrow \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$  be the map given by  $\Delta$  as in 13.1. Let  $\mathcal{P}$  be given.

Write  $A = \overline{\bigcup_{n=1}^\infty A_n}$ , where  $A_n = A(W, \alpha_n) \oplus W_n$  as in section 6. Without loss of generality, we may assume  $\mathcal{F} \subset A_{N'}$  for some integer  $N'$  and  $\mathcal{P} \subset [\iota'](\mathcal{P}_{N'})$  for some finite subset  $\mathcal{P}_{N'} \subset \underline{K}(A_{N'})$ , where  $\iota' : A_{N'} \rightarrow A$  is the embedding.

Let  $\delta_0 > 0$  and let  $\mathcal{G}_0 \subset A_{N'}$  be a finite subset satisfying the following:  $\text{Bott}(L, w)|_{\mathcal{P}}$  is well defined for any  $\mathcal{G}_0$ - $\delta_0$ -multiplicative completely positive contractive linear map  $L : A \rightarrow C$  and any unitary  $w \in \tilde{C}$  with  $\|[L(g), w]\| < 2\delta_0$  for all  $g \in \mathcal{G}_0$ . Moreover, if  $w'$  is another unitary, we also require that

$$\text{Bott}(L, ww')|_{\mathcal{P}} = \text{Bott}(L, w)|_{\mathcal{P}} + \text{Bott}(\varphi, w')|_{\mathcal{P}}, \quad (\text{e 13.151})$$

when  $\|[L(g), w']\| < 2\delta_0$  for all  $g \in \mathcal{G}_0$ .

Let  $\varphi$  and  $u$  be given satisfying the assumption for the above  $\mathcal{G}_0$  and  $\delta_0$ .

Now fix  $\varepsilon > 0$  and a finite subset  $\mathcal{F} \subset A$ .

Let  $\varepsilon_1 = \min\{\delta_0/4, \varepsilon/16\}$  and  $\mathcal{F}_1 = \mathcal{F} \cup \mathcal{G}_0$ . Let  $\delta_1 > 0$  (in place of  $\delta$ ),  $\gamma > 0$ ,  $\eta > 0$ ,  $\mathcal{G}_1 \subset A$  (in place of  $\mathcal{G}$ ) be a finite subset,  $\mathcal{P}_1 \subset \underline{K}(A)$  (in place  $\mathcal{P}$ ) be a finite subset,  $\mathcal{U} \subset U(\tilde{A})$  be a finite subset,  $\mathcal{H}_1 \subset A_+ \setminus \{0\}$  be a finite subset, and  $\mathcal{H}_2 \subset A_{s.a.}$  be a finite subset required by 5.3 for the above  $T$  (and for  $\mathbf{T}(n, k) = n$  as  $K_0(\tilde{B}_1)$  is weakly unperforated).

Without loss of generality, we may assume that  $\mathcal{P}_1 \subset [\iota](\mathcal{P}_N)$  for some finite subset  $\mathcal{P}_N \subset \underline{K}(A_N)$ , where  $N \geq N'$  and  $\iota : A_{N'} \rightarrow A$  is the embedding. We assume that  $\delta_1 < \delta_0$ . Without loss of generality, by choosing large  $N$ , we may assume that  $\mathcal{G}_1 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \subset (A_N)_+^1$ . We may also assume that  $\mathcal{U} \subset U(\tilde{A}_N)$ . Write  $w = \lambda_w + \alpha(w)$ , where  $\lambda_w \in \mathbb{T} \subset \mathbb{C}$  and  $\alpha(w) \in A_N$ . As in the remark of 5.3, we may assume that  $[w] \neq 0$  and  $[w] \in \mathcal{P}_N$  for all  $w \in \mathcal{U}$ . Let  $G_u$  be the subgroup generated by  $\{\bar{w} : w \in \mathcal{U}\}$ . We may view  $G_u \subset J_c(K_1(A))$  (see the statement of 13.2). Moreover, for any  $\mathcal{G}_1$ - $\delta_1$ -multiplicative completely positive contractive linear map  $L'$  from  $A_N$  to a non-unital  $C^*$ -algebra  $C'$  induces a homomorphism  $\lambda' : G_u \rightarrow U(\tilde{C})/CU(\tilde{C})$  (see 14.5 of [33]). Furthermore, since  $K_i(A_N)$  is finitely generated,  $i = 0, 1$ , we may assume, with even smaller  $\delta_1$  and larger  $\mathcal{G}_1$ , that  $[\Phi_{w, L'}]$  defines an element in  $KL(C(\mathbb{T}, \tilde{A}_N), C)$ , if  $\|[L'(g), w]\| < \delta_1$  for all  $g \in \mathcal{G}_1$ .

Set  $\mathcal{G} = \mathcal{F}_1 \cup \mathcal{G}_1 \cup \{\alpha(w) : w \in \mathcal{U}\}$  and set a rational number

$$0 < \sigma_0 < \min\{\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}, \gamma/4\}.$$

Choose  $\delta = \min\{\varepsilon_1/16, \delta_1/16, \gamma/16, \eta/16\}$ . We may write  $u = 1_{\tilde{B}} + \alpha(u)$ , where  $\alpha(u) \in B$ . Since  $B \otimes Q \cong B$ ,  $K_i(B)$  is divisible ( $i = 0, 1$ ). Therefore  $KL(A, B) = \text{Hom}(K_*(A), K_*(B))$  and there is  $\kappa \in KL(C(\mathbb{T}, \tilde{A}_N), \tilde{B})$  such that

$$[\Phi_{u, \varphi \circ \iota}]|_{\mathcal{P}_{N'} \cup \beta(\mathcal{P}_{N'})} = \kappa|_{\mathcal{P}_{N'} \cup \beta(\mathcal{P}_{N'})} \quad \text{and} \quad [u] = \kappa([z \otimes 1_{\tilde{A}_N}]). \quad (\text{e 13.152})$$

Note that  $B \cong B \otimes \mathcal{Z}_0$ . Define  $\psi_{b, w} : B \otimes \mathcal{Z}_0 \rightarrow B \otimes W$  by letting  $\psi_{b, w}(b \otimes a) = b \otimes \psi_{z, w}(a)$  for all  $b \in B$  and  $a \in \mathcal{Z}_0$ , where  $\psi_{z, w} : \mathcal{Z}_0 \rightarrow W$  is a homomorphism defined in 7.11. Note also  $W \otimes Q \cong Q$ . There is a homomorphism  $\psi_{\sigma, W} : W \rightarrow W$  such that  $d_{t_W}(\psi_{\sigma, W}(e_W)) = 1 - \sigma_0$  and

$$t_W(\psi_{\sigma}(a)) = (1 - \sigma_0)t_W(a) \quad \text{for all } a \in W. \quad (\text{e 13.153})$$

Let  $\psi_{w, z}$  be as in 7.11. Note that  $t_W = t_Z \circ \psi_{w, z}$  and  $t_Z = t_W \circ \psi_{z, w}$ , where  $t_W$  and  $t_Z$  are tracial states of  $W$  and  $\mathcal{Z}_0$ , respectively. Let  $\psi_{b, \sigma} : B \rightarrow B$  be defined by  $\psi_{b, \sigma}(b \otimes a) = b \otimes \psi_{w, z} \circ \psi_{\sigma} \circ \psi_{z, w}(a)$  for all  $b \in B$  and  $a \in \mathcal{Z}_0$ . Note that  $\psi_{b, \sigma}(B)^\perp \neq \{0\}$ .

Let  $\varphi_\sigma = \psi_{b, \sigma} \circ \varphi$  and  $\alpha(u_\sigma) = \psi_{b, \sigma}(\alpha(u))$ . Then, by (e 13.153)

$$|\tau \circ \varphi(a) - \tau \circ \varphi_\sigma(a)| \leq (1 - \sigma_0)|\tau(a)| \quad \text{for all } a \in A. \quad (\text{e 13.154})$$

In particular,

$$\tau \circ \psi_\sigma(h) \geq (1 - \sigma_0)\tau(\varphi(h)) \geq (1 - \sigma_0)\Delta_1(\hat{h}) \text{ for all } h \in (A_+)^1 \setminus \{0\}. \quad (\text{e 13.155})$$

Choose two mutually orthogonal non-zero positive elements  $e_1, e_2 \in \psi_{b,\sigma}(B)^\perp$ . Note that

$$\sum_{i=1}^2 \tau(e_i) < \sigma_0 \text{ for all } \tau \in T(B). \quad (\text{e 13.156})$$

Consider  $C^*$ -algebra  $C_0 = C(\mathbb{T}, \tilde{A}_N)^\circ$ . Since  $Q \otimes W \cong W$ , it is easy to see that  $C(\mathbb{T}, \tilde{A})^\circ$  satisfies the condition in 8.3. It follows from 9.8 that there exists an asymptotic completely positive contractive linear maps  $L_n : C_0 \rightarrow B \otimes M_{k(n)}$  such that

$$[L_n^\sim]_{\mathcal{P} \cup \beta(\mathcal{P})} = \kappa^\otimes|_{\mathcal{P} \cup \beta(\mathcal{P} \cup \{[1_{\tilde{C}}]\})}, \quad (\text{e 13.157})$$

where  $k(n) \rightarrow \infty$  and where

$$\kappa^\otimes|_{\underline{K}(A_N)} = \kappa|_{\underline{K}(A_N)} \text{ and } \kappa^\otimes|_{\beta(\underline{K}(\tilde{A}_N))} = -\kappa|_{\beta(\underline{K}(\tilde{A}_N))}. \quad (\text{e 13.158})$$

In particular,  $\kappa^\otimes(\beta([1_{\tilde{A}_N}])) = -\kappa(\beta([1_{\tilde{A}_N}])) = -[u]$ . For each  $n$ , there are two sequences of completely positive contractive linear maps  $\psi_{0,m} : B \otimes M_{k(n)} \rightarrow B_{0,m} \subset B \otimes M_{k(n)}$  and  $\psi_{1,m} : B \otimes M_{k(n)} \rightarrow D_m \subset B \otimes M_{k(n)}$  such that

$$\lim_{m \rightarrow \infty} \|x - \text{diag}(\psi_{0,m}(x), \psi_{1,m}(x))\| = 0 \text{ for all } x \in B \otimes M_{k(n)}, \quad (\text{e 13.159})$$

$$\lim_{m \rightarrow \infty} \|\psi_{i,m}(ab) - \psi_{i,m}(a)\psi_{i,m}(b)\| = 0 \text{ for all } a, b \in B \otimes M_{k(n)}, \quad i = 0, 1, \quad (\text{e 13.160})$$

$$\lim_{m \rightarrow \infty} \sup\{d_\tau(e_{b,m}) : \tau \in T(B)\} = 0, \quad (\text{e 13.161})$$

where  $e_{b,m} \in B_{0,m}$  is a strictly positive element of  $B_{0,m}$ ,  $D_m \in C_0^0$ ,  $B_{0,m} \perp D_m$ . Since  $K_i(D_m) = \{0\}$ ,  $i = 0, 1$ , by choosing sufficiently large  $n$  and  $m$ , put  $L'_n = \psi_{0,m} \circ L_n$ , we may assume that  $L'_n$  is  $\mathcal{G}$ - $\delta/2$ -multiplicative (with embedding  $\iota : C_0 \rightarrow C(\mathbb{T}, \tilde{A})^\circ$ ) and

$$[L'_n \circ \iota]_{\mathcal{P} \cup \beta(\mathcal{P} \cup \{[1_{\tilde{A}_N}]\})} = \kappa^\otimes|_{\mathcal{P} \cup \beta(\mathcal{P} \cup \{[1_{\tilde{A}_N}]\})}. \quad (\text{e 13.162})$$

Moreover, by (e 13.161), we may assume that  $e_{b,m} \lesssim e_{0,1}$ , where  $e_{0,1} \in B$ ,  $e_{0,1}e_1 = e_1e_{0,1} = e_{0,1}$ . Since  $B$  has almost stable rank one, there is a unitary  $w_1 \in \tilde{B}$  such that  $\text{Ad } w_1 \circ L'_n(a) \in B_{0,e} = \overline{e_1 B e_1}$  for all  $a \in A$ . Put  $L''_n = \text{Ad } w_1 \circ L'_n$ . Let  $u_0 \in \widetilde{B_{0,e}}$  such that  $u_0 = 1_{B_{0,e}} + \alpha(u_0)$  for some  $\alpha(u_0) \in (B_{0,e})_{s.a.}$  and

$$\|L''_n(z \otimes 1_{\tilde{A}_N}) - u_0\| < \delta/16. \quad (\text{e 13.163})$$

Define  $L : A \rightarrow B$  by (for some sufficiently large  $n$  as specified above)

$$L(a) = \text{diag}(L''_n(a), \psi_{b,\sigma} \circ \varphi(a)) \text{ for all } a \in A. \quad (\text{e 13.164})$$

It is ready to check that  $L$  is  $\mathcal{G}$ - $\delta$ -multiplicative. Let  $\lambda' : G_u \rightarrow U(\tilde{B})/CU(\tilde{B})$  be a homomorphism induced by  $L$ . Let  $\lambda = \varphi^\dagger|_{\tilde{A}} - \lambda'$ . Since  $\psi_{b,\sigma} \circ \varphi$  factors through  $B \otimes W$ ,  $[\psi_{b,\sigma} \circ \varphi] = 0$ . the map  $\lambda$  maps  $G_u$  into  $U_0(\tilde{B})/CU(\tilde{B})$ . Since  $U_0(\tilde{B})/CU(\tilde{B})$  is divisible, we may extend  $\lambda$  to a map from  $J_c(K_1(A))$  into  $\text{Aff}(T(\tilde{B}))/\mathbb{Z}$ . Choose a non-zero element  $e_0 \in B$  with  $e_0e_2 = e_2e_0 = e_0$  such that  $d_\tau(e_0)$  is continuous on  $T(B)$ . Let  $\lambda_T : T(\overline{e_0 B e_0}) \rightarrow T(A)$  be an affine continuous map defined by  $\lambda_T(t) = \tau_A$  for all  $t \in T(\overline{e_0 B e_0})$ , where  $t_A$  is a fixed trace in  $T(A)$ . Define  $\lambda_{cu} : U(\tilde{A})/CU(\tilde{A}) \rightarrow U_0(\overline{e_0 B e_0})/CU(\overline{e_0 B e_0})$  by  $\lambda_{cu}|_{J_c(K_1(\tilde{A}))} = \lambda$  and  $\lambda_{cu}|_{U_0(\tilde{A})/CU(\tilde{A})} = \lambda_T^\sharp$ ,



i.e.,  $\lambda_{cu}(f)(t) = f(\lambda_T(t))$  for all  $t \in T(\overline{e_0 B e_0})$ . Define  $\lambda_K : \underline{K}(A) \rightarrow \underline{K}(\overline{e_0 B e_0})$  by  $\lambda_K = 0$ . Then  $(\lambda_J, \lambda_{cu}, \lambda_T)$  is compatible. It follows from 11.8 that there exists a homomorphism  $\varphi_{cu} : A \rightarrow \overline{e_0 B e_0}$  such that  $([\varphi_{cu}], \varphi_{cu}^\dagger, (\varphi_{cu})_T) = (\lambda_K, \lambda_{cu}, \lambda_T)$ .

Now define  $\Phi : A \rightarrow B$  by  $\Phi(a) = \text{diag}(\varphi_{cu}(a), L(a))$ . Then  $\Phi$  is  $\mathcal{G}_1$ - $\delta$ -multiplicative,

$$\tau \circ \Phi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \quad (\text{by (e 13.155)}) \quad (\text{e 13.165})$$

$$\|\tau \circ \Phi(h) - \tau \circ \varphi(h)\| < \gamma \text{ for all } h \in \mathcal{H}_2, \quad (\text{e 13.166})$$

$$[\Phi]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}} \text{ and} \quad (\text{e 13.167})$$

$$\Phi^\dagger(\bar{w}) = \lambda(\bar{w}) + \lambda'(\bar{w}) = \varphi^\dagger(\bar{w}) \text{ for all } w \in \mathcal{U}. \quad (\text{e 13.168})$$

By (e 13.165),  $\Phi$  is also  $T\text{-}\mathcal{H}_1$ -full. By applying 5.3, we obtain a unitary  $W \in \tilde{B}$  such that

$$\|W^* \Phi(f) W - \varphi(f)\| < \varepsilon_1 \text{ for all } f \in \mathcal{F} \cup \mathcal{G}_0. \quad (\text{e 13.169})$$

Let  $v = W^*(1_{\tilde{B}} + \alpha(u_0) + \psi_{b,\sigma}(\alpha(u)))W$ . Then  $v$  is a unitary. We have

$$\|[\varphi(f), v]\| < \varepsilon_1 + \delta \text{ for all } f \in \mathcal{F} \cup \mathcal{G}_0. \quad (\text{e 13.170})$$

We then compute that

$$\text{Bott}(\varphi, uv)|_{\mathcal{P}} = \text{Bott}(\varphi, u)|_{\mathcal{P}} + \text{Bott}(\varphi, v)|_{\mathcal{P}} = 0 \text{ and } [uv] = 0. \quad (\text{e 13.171})$$

□

**Remark 13.7.** Lemma 13.6 still holds by replacing  $Q$  by any UHF-algebra of infinite type if  $K_i(A)$  is finitely generated.

**Lemma 13.8.** *Let  $A \in \mathcal{B}_T$  have continuous scale. For any  $\varepsilon > 0$  and any finite subset  $\mathcal{F} \subset A$ , there exists  $\delta > 0$  and a finite subset  $\mathcal{G} \subset A$  satisfying the following: Suppose that  $\varphi : A \rightarrow B \cong B \otimes W$  with continuous scale, where  $B \in \mathcal{D}_0$ , is a homomorphism which maps strictly positive elements to strictly positive elements and suppose that there is a unitary  $u \in \tilde{B}$  such that*

$$\|[\varphi(g), u]\| < \delta \text{ for all } g \in \mathcal{G}. \quad (\text{e 13.172})$$

*Then there exists a continuous path of unitaries  $\{u(t) : t \in [0, 1]\} \subset \tilde{B}$  such that  $u(0) = u$ ,  $u(1) = 1_{\tilde{B}}$  and*

$$\|[\varphi(f), u(t)]\| < \varepsilon \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.173})$$

*Proof.* Note that, by 12.4,  $A \cong A \otimes \mathcal{Z}_0$ . We identify  $A$  with  $A \otimes \mathcal{Z}_0$ . Let  $\psi_{z,w} : \mathcal{Z}_0 \rightarrow W$  be defined in 7.11. Let  $\psi_{w,a} : A \otimes W \rightarrow A \otimes \mathcal{Z}_0$  defined by  $\psi_{w,a}(a \otimes w) = a \otimes \psi_{w,z}(w)$  for all  $a \in A$ ,  $z \in \mathcal{Z}_0$  and  $w \in W$ . Put  $A_1 = A \otimes W$ . Fix  $\varepsilon > 0$  and a finite subset  $\mathcal{F} \subset A$ .

Note  $T(A) = T(A \otimes W)$  and  $\rho_{\tilde{A}}(K_0(\widetilde{A \otimes W})) = \mathbb{Z}$ . It follows from 11.8 that there exists a homomorphism  $h_{a,w} : A \rightarrow A \otimes W$  such that  $(h_{a,w})_T = \text{id}_{T(A)}$  and  $h_{a,w}^\dagger|_{J_c(K_1(A))} = \bar{1}$  and  $h_{a,w}^\dagger|_{\text{Aff}(T(\tilde{A}))/\mathbb{Z}} = \text{id}_{\text{Aff}(T(\tilde{A}))/\mathbb{Z}}$ .

Let  $\mathcal{F}_1 = h_{a,w}(\mathcal{F})$ . Choose  $\mathcal{G}_w \in A \otimes W$  and  $\delta_w > 0$  which are required by 20.11 of [17] for  $\mathcal{F}_1$  and  $\varepsilon/16$ .

Suppose that  $\psi : A \rightarrow B$  is a homomorphism which maps strictly positive elements to strictly positive elements and suppose that there is a unitary  $v \in \tilde{B}$  such that

$$\|[\psi(g), v]\| < \delta_w/2 \text{ for all } g \in \psi_{w,a}(\mathcal{G}_w) \quad (\text{e 13.174})$$

and suppose  $\psi^\dagger$  maps  $J_c(K_1(A))$  to  $\bar{1}$ .



Consider the homomorphism  $\psi' : A \rightarrow B$  defined by  $\psi' = \psi \circ \psi_{w,a} \circ h_{a,w}$ . Note that  $[\psi'] = [\psi]$  in  $KL(A, B)$  and  $\tau \circ \psi' = \tau \circ \psi$  for all  $\tau \in T(B)$  and  $\psi^\dagger = (\psi')^\dagger$ . Therefore, by 5.3 (and 5.7), there is a unitary  $V \in \tilde{B}$  such that

$$\|V^* \psi'(g)V - \psi(g)\| < \min\{\delta_W/2, \varepsilon/16\} \text{ for all } g \in \psi_{w,a}(\mathcal{G}_w) \cup \mathcal{F}. \quad (\text{e 13.175})$$

Define  $\psi_W : A \otimes W \rightarrow B$  by  $\psi_W = \text{Ad } V \circ \psi \circ \psi_{w,a}$ . Then

$$\|[\psi_W(g), v]\| < \delta \text{ for all } g \in \mathcal{G}_W. \quad (\text{e 13.176})$$

It follows from 20.11 of [17] that there exists a continuous path of unitaries  $\{v(t) : t \in [0, 1]\} \subset U(\tilde{B})$  with  $v(0) = u$  and  $v(1) = 1_{\tilde{B}}$  such that

$$\|[\psi_W(g), v(t)]\| < \varepsilon/16 \text{ for all } g \in \mathcal{F}_1. \quad (\text{e 13.177})$$

Therefore,

$$\|[\text{Ad } V \circ \psi'(f), v(t)]\| < \varepsilon/16 \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.178})$$

It follows from this and (e 13.175) that

$$\|[\psi(f), v(t)]\| < \varepsilon/8 \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.179})$$

Now we consider the general case that  $\psi^\dagger(J_c(K_1(A))) \neq \bar{1}$ . Let

$$\Delta_A(\hat{a}) = \inf\{\tau(a) : \tau \in T(A)\} \text{ for all } a \in A_+^1 \setminus \{0\} \text{ and} \quad (\text{e 13.180})$$

$$\Delta_0(\hat{c}) = \inf\{(\int f dm) \Delta_A(a) : c \geq f \otimes a, f \in C(\mathbb{T}), a \in A\} \quad (\text{e 13.181})$$

for all  $c \in C(\mathbb{T}, \tilde{A})^\circ$ , where  $m$  is the normalized Haar measure on  $\mathbb{T}$ . Put  $\Delta = \Delta_0/2$ .

Put  $A_c = C(\mathbb{T}, \tilde{A})^\circ$ . Let  $\mathcal{H}_1 \subset ((\tilde{A}_c)_+^1 \setminus \{0\})$  be a finite subset,  $\gamma_1 > 0$ ,  $\gamma_2 > 0$ ,  $\delta_c > 0$ ,  $\mathcal{G}_1 \subset \tilde{A}_c$  (in place of  $\mathcal{G}$ ) and  $\mathcal{P} \subset \underline{K}(\tilde{A}_c)$ ,  $\mathcal{H}_2 \subset A_c$  and  $\mathcal{U} \subset J_c(K_1(\tilde{A}_c))$  be finite subsets with  $[\mathcal{U}] \subset \mathcal{P}$  be required by 13.4 for  $\min\{\delta_W/4, \varepsilon/16\}$  (in place of  $\varepsilon$ ) and  $\psi_{a,w}(\mathcal{G}_W)$  (in place of  $\mathcal{F}$ ) and  $\Delta$ . With smaller  $\delta_c > 0$ ,  $\gamma_i$ , without loss of generality, we may assume that  $\mathcal{H}_1 = \{g \otimes 1_{\tilde{A}} : g \in \mathcal{H}_{1,T}\} \cup \{1_{C(\mathbb{T})} \otimes a : a \in \mathcal{H}_{1,A}\}$ , and  $\mathcal{G}_1 = \{g \otimes 1_{\tilde{A}} : g \in \mathcal{G}_{1,T}\} \cup \{1_{C(\mathbb{T})} \otimes a : a \in \mathcal{G}_{1,A}\}$ ,  $\mathcal{H}_2 = \{g \otimes 1_{\tilde{A}} : g \in \mathcal{H}_{2,T}\} \cup \{1_{C(\mathbb{T})} \otimes a : a \in \mathcal{H}_{2,A}\}$ , where  $\mathcal{H}_{1,T}, \mathcal{H}_{2,T}, \mathcal{G}_{1,T} \subset C(\mathbb{T})$ , and  $\mathcal{H}_{1,A}, \mathcal{H}_{2,A}, \mathcal{G}_{1,A} \subset \tilde{A}$  are finite subsets.

Let  $\mathcal{G}' = \mathcal{G}_{1,A} \cup \mathcal{F}$  and  $\delta' = \min\{\delta_c/2, \delta_W/2, \varepsilon/16\}$ . Let  $0 < \delta < \delta'$  and  $\mathcal{G} \supset \mathcal{G}'$  be finite subset such that any  $\mathcal{G}$ - $\delta$ -multiplicative completely positive contractive linear map  $L'$  from  $A$  to a  $C^*$ -algebra  $C$  and any unitary  $u' \in \tilde{C}$  with property  $\|[L'(g), u']\| < 2\delta$  for all  $g \in \mathcal{G}$  gives a  $\mathcal{G}_1$ - $\delta$ -multiplicative completely positive contractive linear map from  $C(\mathbb{T}, \tilde{A})$  to  $\tilde{C}'$ .

Suppose that  $\varphi : A \rightarrow B$  is a homomorphism which maps strictly positive elements to strictly positive elements and  $u \in \tilde{B}$  such that

$$\|[\varphi(g), u]\| < \delta \text{ for all } g \in \mathcal{G}. \quad (\text{e 13.182})$$

Note that  $B \otimes Q \otimes Q \cong B$ . We may assume that  $\varphi(A) \subset B \otimes 1_Q \otimes 1_Q$  and  $u \in \tilde{B} \otimes 1_Q \otimes 1_Q$ . Let  $\{e_n\}$  be an approximate identity for  $A$ . Consider  $v_n = u(\exp(ie_n \otimes h))$ , where  $h \in Q \otimes 1_Q$  with  $\text{sp}(h) = [-\pi, \pi]$  and  $t_Q(h) = 0$  and where  $t_Q$  is the tracial state of  $Q$ . Let  $p_k, q_{1,k}, q_{2,k} \in 1_Q \otimes Q$  be mutually orthogonal projections with  $t_U(p_k) = 1 - 1/k$ ,  $q_{i,k} = 1/2k$ ,  $i = 1, 2$ , and  $p_k \oplus q_{1,k} \oplus q_{2,k} = 1_Q \otimes 1_Q$ ,  $k = 1, 2, \dots$ . Put  $B_k = B \otimes p_k$ ,  $B_{i,k} = B \otimes q_{i,k}$ ,  $i = 1, 2$ ,  $k = 1, 2, \dots$

By 11.8, there are homomorphisms  $\Phi_{i,k} : A \rightarrow B_{i,k}$  such that  $\tau(\Psi_{i,k}(a)) = (1/2k)\tau(\varphi(a))$  for all  $a \in A$  and

$$\Psi_{1,k}^\dagger|_{J_c(K_1(A))} = -(1 - \frac{1}{k})\varphi^\dagger|_{J_c(K_1(A))} \text{ and } \Psi_{2,k}^\dagger|_{J_c(K_1(A))} = (1 - \frac{1}{k})\varphi^\dagger|_{J_c(K_1(A))}, \quad (\text{e 13.183})$$

$k = 1, 2, \dots$ . Define  $\psi'_{n,k} : A \rightarrow C_k := B_k \oplus B_{1,k}$  by  $\psi'_{n,k}(a) = \varphi(a) \otimes p_k \oplus \Psi_{1,k}(a)$  for all  $a \in A$ , and define  $\psi_{n,k} : A \rightarrow B \otimes 1_Q \otimes 1_Q$  by  $\psi_{n,k}(a) = \psi'_{n,k}(a) \oplus \Psi_{2,k}(a)$  for all  $a \in A$ ,  $k = 1, 2, \dots$ . Write  $v_n = \lambda + \alpha(v_n)$  for some  $\lambda \in \mathbb{T}$  and  $\alpha(v_n) \in B \otimes 1_Q \otimes 1_Q$ . Let  $v_{n,k} = \lambda \cdot 1_{\tilde{C}_k} + \alpha(v_n)(p_k \oplus q_{1,k})$  and  $w_{n,k} = \lambda \cdot 1_{\tilde{B}} + \alpha(v_n)(p_k \oplus q_{1,k})$ . Choose a completely positive contractive linear map  $L_{n,k} = \Phi_{w_{n,k}, \psi_k} : C(\mathbb{T}, \tilde{A})^o \rightarrow B \otimes Q \otimes Q$  induced by the unitary  $w_{n,k}$  and  $\psi_k$ . Let  $\Phi_{u,\varphi} : C(\mathbb{T}, \tilde{A})^o \rightarrow B \otimes Q \otimes Q$  be induced by  $v_n$  and  $\varphi$ .

Note that  $U(\tilde{B})/CU(\tilde{B}) = \text{Aff}(T(\tilde{B}))/\mathbb{Z}$ . By applying 20.9 of [17], for all sufficiently large  $n$  and  $k$  (we then fix a pair  $n$  and  $k$ )

$$\tau(L_{n,k}(h)) \geq \Delta_0(\hat{h})/2 = \Delta(\hat{h}) \text{ for all } \tau \in T(B) \text{ and for all } h \in \mathcal{H}_1, \quad (\text{e 13.184})$$

$$|\tau(L_{n,k}(h)) - \tau(\Phi_{v_n,\varphi})(h)| < \gamma_1 \text{ for all } h \in \mathcal{H}_2 \text{ and } \quad (\text{e 13.185})$$

$$\text{dist}(L_{n,k}^\dagger(\bar{w}), \Phi_{v_n,\varphi}^\dagger(\bar{w})) < \gamma_2 \text{ for all } w \in \mathcal{U}. \quad (\text{e 13.186})$$

It follows from 13.4 that there exists a unitary  $U \in \widetilde{B \otimes Q \otimes Q}$  such that

$$\|U^* \psi_{n,k}(g)U - \varphi(g)\| < \min\{\delta_W/4, \varepsilon/16\} \text{ for all } g \in \psi_{w,a}(\mathcal{G}_W) \text{ and } \quad (\text{e 13.187})$$

$$\|U^* w_{n,k}U - v_n\| < \min\{\delta_W/4, \varepsilon/16\}. \quad (\text{e 13.188})$$

Now consider  $\text{Ad } U \circ \psi'_{n,k} : A \rightarrow D_k := U^* C_k U$  and the unitary  $U^* v_{n,k} U \in \tilde{D}_k$ . Note, by (e 13.183),  $(\text{Ad } U \circ \psi'_{n,k})^\dagger|_{J_c(K_1(A))} = \bar{1}$ . So we reduce this case to the case that has been proved. Thus there is a continuous path of unitaries  $\{V(t) : t \in [2/3, 1]\} \subset \tilde{D}_k$  such that  $V(2/3) = U^* v_{n,k} U$  and  $V(1) = 1_{\tilde{D}_k}$  and

$$\|[\text{Ad } U \circ \psi'_{n,k}(f), V(t)]\| < \varepsilon/8 \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.189})$$

Note that  $U^* w_{n,k} U = \lambda \otimes 1_{\tilde{B}} + U^* \alpha(v_{n,k}) U$ . Write  $V(t) = \lambda(t) \cdot 1_{\tilde{D}_k} + \alpha(V(t))$  for some  $\lambda(t) \in \mathbb{T}$  and  $\alpha(V(t)) \in D_k$ . Put  $Z(t) = \lambda(t) \cdot 1_{\tilde{B}} + \alpha(V(t))$ . Then  $Z(2/3) = U^* w_{n,k} U$  and  $Z(1) = 1_{\tilde{B}}$ . Since  $B_{2,k} \perp C_k$ , we have that

$$\|[\text{Ad } U \circ \psi_{n,k}(g), Z(t)]\| < \varepsilon/8 \text{ for all } f \in \mathcal{F}. \quad (\text{e 13.190})$$

By (e 13.188), we may write  $v_n^* U^* w_{n,k} U = \exp(ib)$  for some  $b \in \tilde{B}_{s.a.}$  with  $\|b\| \leq 2 \arcsin(\varepsilon/32)$ . Define  $Z(t) = v_n \exp(i(3(t - 1/3)b))$  for  $t \in [1/3, 2/3]$ . Then  $Z(1/3) = v_n$ . We also have

$$\|[\text{Ad } U \circ \psi_{n,k}(g), Z(t)]\| < \varepsilon/8 \text{ for all } t \in [1/3, 1]. \quad (\text{e 13.191})$$

It follows that

$$\|[\varphi(g), Z(t)]\| < \varepsilon/8 + \varepsilon/16 \text{ for all } t \in [1/3, 1]. \quad (\text{e 13.192})$$

Define  $Z(t) = u(\exp(3ite_n \otimes h))$  for  $t \in [0, 1/3]$ . Then  $Z(0) = u$  and  $\{Z(t) : t \in [0, 1]\}$  is a continuous path of unitaries in  $\tilde{B}$ . Moreover,

$$\|[\varphi(g), Z(t)]\| < \varepsilon \text{ for all } g \in \mathcal{F} \text{ and } t \in [0, 1]. \quad (\text{e 13.193})$$

□

**Theorem 13.9.** *Let  $A \in \mathcal{B}_T$  have continuous scale. Let  $\mathcal{P} \subset \underline{K}(A)$  be a finite subset, let  $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$  be projections of  $M_s(\tilde{A})$  (for some integer  $s \geq 1$ ) such that  $\{[p_1] - [q_1], [p_2] - [q_2], \dots, [p_k] - [q_k]\} \subset \mathcal{P}$  generates a free subgroup  $G_{u0}$  of  $K_0(A)$ , let  $\sigma > 0$ ,  $\varepsilon_0 > 0$  and  $\mathcal{F}_0 \subset A$  be a finite subset. There exist  $\delta_0 > 0$  and  $\mathcal{G}_0 \subset A$  such that the following hold: For any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subset A$ , any homomorphism  $\varphi : A \rightarrow B = B_1 \otimes Q$  which maps strictly positive elements to strictly positive elements, where  $B_1 \cong B_1 \otimes \mathcal{Z}_0 \in \mathcal{D}_0$  has continuous scale, and any unitary  $u \in U(\tilde{B})$  such that*

$$\|[\varphi(g), u]\| < \delta_0 \text{ for all } g \in \mathcal{G}_0, \quad (\text{e 13.194})$$

*there exists a continuous path of unitaries  $\{v(t) : t \in [0, 1]\} \subset U(\tilde{B})$  such that*

$$\|[\varphi(g), v(0)]\| < \varepsilon \text{ for all } g \in \mathcal{G}_0 \cup \mathcal{F}, \quad (\text{e 13.195})$$

$$\|[\varphi(f), v(t)]\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}_0, \quad (\text{e 13.196})$$

$$\text{Bott}(\varphi, uv(1))|_{\mathcal{P}} = 0, \quad [uv(1)] = 0 \text{ and} \quad (\text{e 13.197})$$

$$\text{dist}(\overline{[(1_s - \varphi(p_i)) + (uv(1))_s \varphi(p_i)](1_s - \varphi(q_i)) + (uv(1))_s^* \varphi(q_i)}], \bar{1}) < \sigma, \quad (\text{e 13.198})$$

where  $1_s = 1_{M_s}$  and  $(uv(1))_s = uv(1) \otimes 1_{M_s}$ .

*Proof.* Define  $\Delta_1(\hat{h}) = \inf\{\tau(h) : \tau \in T(A)\}$  for  $h \in A_+^1 \setminus \{0\}$ . Let  $\Delta = \Delta_1/2$ . Let  $T : A_+^1 \setminus \{0\} \rightarrow \mathbb{R}_+ \setminus \{0\} \times \mathbb{N}$  be the map given by  $\Delta$  as in 13.1. Let  $\varepsilon_0, \sigma, \mathcal{F}_0, \mathcal{P}$  and  $\{p_1, \dots, p_k, q_1, q_2, \dots, q_k\} \subset M_s(\tilde{A})$  be given.

Write  $p_l = (a_{i,j}^{p_l})_{s \times s}$  and  $q_l = (a_{i,j}^{q_l})_{s \times s}$ , where  $a_{i,j}^{p_l}, a_{i,j}^{q_l} \in \tilde{A}$ ,  $1 \leq i, j \leq s$ ,  $1 \leq l \leq k$ . Let  $\mathcal{F}_p$  be a finite subset in  $A$  such that  $a_{i,j}^{p_l}, a_{i,j}^{q_l} \in \mathbb{C} \cdot 1 + \mathcal{F}_p$ .

In what follows, if  $L' : A \rightarrow C'$  is a map, we will continue to use  $L'$  for  $L'^{\sim} : \tilde{A} \rightarrow \tilde{C}'$  and  $L' \otimes \text{id}_{M_s}$  as well as  $L'^{\sim} \otimes \text{id}_{M_s}$  when it is convenient. Moreover,  $1_s := 1_{M_s}$ .

Let  $\delta'_0 > 0$  and let  $\mathcal{G}'_0 \subset A$  be a finite subset satisfying the following:  $\text{Bott}(L, w)|_{\mathcal{P}}$  is well defined for any  $\mathcal{G}'_0$ - $\delta'_0$ -multiplicative completely positive contractive linear map  $L : A \rightarrow C$  and any unitary  $w \in C$  with  $\|[L(g), w]\| < 2\delta'_0$  for all  $g \in \mathcal{G}_0$ . Moreover, if  $w'$  is another unitary, we also require that

$$\text{Bott}(L, ww')|_{\mathcal{P}} = \text{Bott}(L, w)|_{\mathcal{P}} + \text{Bott}(\varphi, w')|_{\mathcal{P}}, \quad (\text{e 13.199})$$

when  $\|[L(g), w']\| < \delta'_0$  for all  $g \in \mathcal{G}'_0$ . Moreover, for any  $\mathcal{G}'_0$ - $\delta'_0$ -multiplicative completely positive contractive linear map  $L'$  from  $A$  to a non-unital  $C^*$ -algebra  $C'$  induces a homomorphism  $\lambda' : G_u \rightarrow U(\tilde{C})/CU(\tilde{C})$  (see 14.5 of [33]). Furthermore, using 14.5 of [33] again, we assume that, for any unitary  $w' \in M_s(\tilde{C})$  with the property that  $\|[L'(g), w']\| < 2\delta'_0$  for all  $g \in \mathcal{G}'_0$ ,  $\Phi_{w', L'}$  induces a homomorphism  $\lambda_{L', w'}$  from  $G_{u0}$  to  $U(\tilde{C})/CU(\tilde{C})$  and, for  $1 \leq i \leq k$ ,

$$\text{dist}(\overline{[(1_s - L'(p_i)) + w'_s L'(p_i)](1_s - L'(q_i)) + (w'_s)^* L'(q_i)}], \lambda_{L', w'}([p_i] - [q_i])) < \sigma/64, \quad (\text{e 13.200})$$

where  $w'_s = w' \otimes 1_s$ . We may assume that  $\delta'_0$  is smaller than  $\delta_0$  in 13.6 and  $\mathcal{G}'_0$  is larger than  $\mathcal{G}_0$  in 13.6 for the above  $\mathcal{P}$ .

Let  $\delta_W > 0$  and let  $\mathcal{G}_W \subset A$  be finite subset required by 13.8 for  $\min\{\varepsilon_0/4, \delta'_0/2\}$  (in place of  $\varepsilon$ ) and  $\mathcal{F}_0 \cup \mathcal{G}'_0$ . Put  $\delta''_0 = \min\{\delta'_0/4, \delta_W/4\}$  and  $\mathcal{G}''_0 = \mathcal{G}'_0 \cup \mathcal{G}_W \cup \mathcal{F}_0 \cup \mathcal{F}_p$ .

Let  $\varepsilon_1 = \min\{\delta''_0/4, \varepsilon_0/16, \sigma/16\}/2^{10}(s+1)^2$ . Let  $\delta_1 > 0$  (in place of  $\delta$ ),  $\gamma > 0$ ,  $\eta > 0$ ,  $\mathcal{G}_1 \subset A$  (in place of  $\mathcal{G}$ ) be a finite subset,  $\mathcal{P}_1 \subset \underline{K}(A)$  (in place  $\mathcal{P}$ ) be a finite subset,  $\mathcal{U} \subset U(\tilde{A})$  be a finite subset,  $\mathcal{H}_1 \subset A_+ \setminus \{0\}$  be a finite subset, and  $\mathcal{H}_2 \subset A_{s.a.}$  be a finite subset required by 5.3 for  $\varepsilon_1$  (in place of  $\varepsilon$ ) and  $\mathcal{G}''_0$  (in place of  $\mathcal{F}$ ) the above  $T$  (and  $\mathbf{T}(n, k) = n$ ).

We assume that  $\delta_1 < \delta''_0$  and that  $\mathcal{G}_1 \cup \mathcal{H}_1 \cup \mathcal{H}_2 \subset (A)_+^1$ . Write  $w = \lambda_w + \alpha(w)$ , where  $\lambda_w \in \mathbb{T} \subset \mathbb{C}$  and  $\alpha(w) \in A$ . As in the remark of 5.3, we may assume that  $[w] \neq 0$  and  $[w] \in \mathcal{P}$

for all  $w \in \mathcal{U}$ . Let  $G_u$  be the subgroup generated by  $\{\bar{w} : w \in \mathcal{U}\}$ . We may view  $G_u \subset J_c(K_1(A))$  (see the statement of 13.2).

Note that  $B \cong B \otimes \mathcal{Z}_0$ . Define  $\psi_{b,W} : B \otimes \mathcal{Z}_0 \rightarrow B \otimes W$  by letting  $\psi_{b,W}(b \otimes a) = b \otimes \psi_{z,w}(a)$  for all  $b \in B$  and  $a \in \mathcal{Z}_0$ , where  $\psi_{z,w} : \mathcal{Z}_0 \rightarrow W$  is a homomorphism defined in 7.11. Note that, by [17],  $B \otimes W \in \mathcal{M}_0$  with continuous scale.

Set  $\mathcal{G}_2 = \mathcal{G}_0 \cup \mathcal{G}_1 \cup \{\alpha(w) : w \in \mathcal{U}\}$  and set a rational number

$$0 < \sigma_0 < \min\{\inf\{\Delta(\hat{h}) : h \in \mathcal{H}_1\}, \gamma/4\}.$$

Without loss of generality, we may assume that there is  $e_A, e'_A \in A_+$  with  $\|e'_A\| = \|e_A\| = 1$  such that

$$e_A g = g e_A = g \text{ for all } g \in \mathcal{G}_2 \text{ and } e'_A e_A = e_A. \quad (\text{e 13.201})$$

Choose a pair of mutually orthogonal non-zero positive elements  $e_0, e'_0 \in (\overline{e'_A e_A})^\perp$  such that

$$d_\tau(e_0 + e'_0) < \sigma_0 \text{ for all } \tau \in T(A). \quad (\text{e 13.202})$$

Choose an integer  $K \geq 1$  such that  $1/K < \min\{\sigma_0/4, \inf\{d_\tau(e_0) : \tau \in T(A)\}\}$  and choose  $\delta_0 = \min\{\varepsilon_1/16, \delta_1/16, \gamma/16, \eta/16\}/64(s+1)^3(K+1)^2$ . Put  $\mathcal{G}_0 = \mathcal{G}_2 \cup \{e_A, e'_A, e_0, e'_0\}$ .

Now let  $\varphi$  and  $u$  be given satisfying the assumption for the above  $\mathcal{G}_0$  and  $\delta_0$ . Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be a finite subset. We may write  $u = 1_{\tilde{B}} + \alpha(u)$ , where  $\alpha(u) \in B$ . Put  $\mathcal{Q} = \mathcal{P} \cup \beta(\mathcal{P})$ .

Note also  $W \otimes Q \cong W$ . Let  $e_q \in Q$  be a projection with  $t_U(e_q) = 1/K$ , where  $t_Q$  is the tracial state of  $Q$ . Define  $\psi_{1/K,W} : W \rightarrow W \otimes Q$  by  $\psi_{1/K,W}(a) = a \otimes e_q$  for all  $a \in A$ . Then

$$t_W(\psi_{1/K,W}(a)) = (1/K)t_W(a) \text{ for all } a \in W. \quad (\text{e 13.203})$$

Let  $\psi_{w,z}$  be as in 7.11. Note that  $t_W = t_Z \circ \psi_{w,z}$  and  $t_Z = t_W \circ \psi_{z,w}$ , where  $t_W$  and  $t_Z$  are tracial states of  $W$  and  $\mathcal{Z}_0$ , respectively. Let  $\psi_{b,1/K} : B \rightarrow B$  be defined by  $\psi_{b,1/K}(b \otimes a) = b \otimes \psi_{w,z} \circ \psi_{1/K,W} \circ \psi_{z,w}(a)$  for all  $b \in B$  and  $a \in \mathcal{Z}_0$ . Let  $\psi_{b,w,1/K} : B \rightarrow B \otimes W \otimes e_q$  be defined by  $\psi_{b,w,1/K}(b \otimes a) = b \otimes \psi_{1/K,W} \circ \psi_{z,w}(a)$  for all  $b \in B$  and  $a \in \mathcal{Z}_0$ .

By applying 13.6, there is a unitary  $v_1 \in \tilde{B}$  such that

$$\|[\varphi(g), v_1]\| < \min\{\delta_2, \varepsilon\} \text{ for all } g \in \mathcal{F} \cup \mathcal{G}_0 \text{ and} \quad (\text{e 13.204})$$

$$\text{Bott}(\varphi, uv_1)|_{\mathcal{P}} = 0 \text{ and } [uv_1] = 0. \quad (\text{e 13.205})$$

Note that

$$\|[\varphi(g), uv_1]\| < \delta_0 + \delta_2 \text{ for all } g \in \mathcal{G}_0. \quad (\text{e 13.206})$$

We may write  $uv_1 = 1_{\tilde{B}} + \alpha(uv_1)$  for some  $\alpha(uv_1) \in B$ . Define  $\psi' : A \rightarrow B$  by  $\psi'(a) = \psi_{b,1/K} \circ \varphi(a)$  for all  $a \in A$ . Using (e 13.202), by replacing  $\psi'$  by  $\text{Ad } w_1 \circ \psi'$  for some unitary  $w_1$ , we may assume that  $\psi'(A) \subset B_0 := e_{0,b} B e_{0,b}$ , where  $e_{0,b} = \varphi(e_0)$ . Let  $v'_2 = 1_{\tilde{B}} + \psi_{b,1/K}(\alpha(uv_1))$ ,  $v_2 = ((v'_2)^*)^K$  and  $v''_2 = 1_{\tilde{B}_0} + \psi_{b,1/K}(\alpha(uv_1))$ . Note that  $[\psi']|_{\mathcal{P}} = 0$ , since it factors through  $B \otimes W$ . Moreover

$$\text{Bott}(\psi', v'_2)|_{\mathcal{P}} = 0 \text{ and } \text{Bott}(\psi', (v'_2)^K)|_{\mathcal{P}} = 0. \quad (\text{e 13.207})$$

Let  $\lambda_{\varphi, uv_1} : G_{u0} \rightarrow U(M_s(\tilde{B}))/CU(M_s(\tilde{B}))$  be the homomorphism induced by  $\varphi$  and  $uv_1$ , via a map  $\Phi_{uv_1, \varphi}$ . Then (e 13.205) implies that  $\lambda_{\varphi, uv_1}$  maps  $G_{u0}$  to  $\text{Aff}(T(\tilde{B}))/\mathbb{Z}$  (see also [19]). Let  $\lambda_{\psi', v'_2} : G_{u0} \rightarrow \text{Aff}(T(\tilde{B}))/\mathbb{Z}$  be the homomorphism induced by  $\Phi_{v'_2, \psi'}$ . Since  $\tau \circ \psi_{b,1/K}(b) = (1/K)\tau(b)$  for all  $b \in B$  and for all  $\tau \in T(B)$ , it is straightforward that we may write

$$\lambda_{\psi', v'_2}([p_i] - [q_i]) = (1/K)\lambda_{\varphi, uv_1}([p_i] - [q_i]), \quad (\text{e 13.208})$$

$i = 1, 2, \dots, k$ . It follows that, by the choice of  $\delta_1$  and  $\delta_2$ , since  $v_2 = ((v'_2)^*)^K$ ,

$$\text{dist}(Z'_i, -(\lambda_{\varphi, uv_1}([\varphi(p_i)] - [\varphi(q_i)]))) < \eta/16, \quad (\text{e 13.209})$$

where  $Z'_i = \overline{[(1_s - \psi'(p_i)) + (v_2)_s \psi'(p_i)]((1 - \psi'(q_i) + (v_2^*)_s \psi'(q_i)))}$ ,  $i = 1, 2, \dots, k$ . As in the proof of 13.6, by applying 11.8, we obtain a homomorphism,  $\psi_{cu} : A \rightarrow \overline{e'_{b,0} B e'_{b,0}}$ , where  $e'_{b,0} = \varphi(e'_0)$ , such that

$$[\psi_{cu}] = 0 \text{ in } KL(A, B) \text{ and } \psi_{cu}^\dagger = -(\psi')^\dagger. \quad (\text{e 13.210})$$

Define  $\psi : A \rightarrow B$  by  $\psi(a) = \psi_{cu}(a) \oplus \psi'(a) \oplus \varphi(e_A a e_A)$  for all  $a \in A$ . Then  $\psi$  is  $\mathcal{G}_2$ - $2\delta_2$ -multiplicative,

$$\tau \circ \psi(h) \geq \Delta(\hat{h}) \text{ for all } h \in \mathcal{H}_1, \quad (\text{e 13.211})$$

$$\|\tau \circ \psi(h) - \tau \circ \varphi(h)\| < \gamma \text{ for all } h \in \mathcal{H}_2, \quad (\text{e 13.212})$$

$$[\psi]|_{\mathcal{P}} = [\varphi]|_{\mathcal{P}} \text{ and } \quad (\text{e 13.213})$$

$$\psi^\dagger(\bar{w}) = -(\psi')^\dagger(\bar{w}) + ((\psi')^\dagger(\bar{w}) + \varphi^\dagger(\bar{w})) = \varphi^\dagger(\bar{w}) \text{ for all } w \in \mathcal{U}. \quad (\text{e 13.214})$$

By (e 13.211),  $\psi$  is  $T\mathcal{H}_1$ -full. By applying 5.3 (as  $K_0(\tilde{B})$  is weakly unperforated), we obtain a unitary  $U \in \tilde{B}$  such that

$$\|U^* \psi(f) U - \varphi(f)\| < \varepsilon_1 \text{ for all } f \in \mathcal{G}'_0. \quad (\text{e 13.215})$$

Let  $v = v_1 U^* (v_2) U$ . Then  $v$  is a unitary. We have

$$\|[\varphi(f), v]\| < \varepsilon_1 + K\delta_2 \text{ for all } f \in \mathcal{G}'_0. \quad (\text{e 13.216})$$

We then compute that, by (e 13.205) and (e 13.207) and the fact that  $\varphi(e_A)v_2 = v_2\varphi(e_A) = \varphi(e_A)$ ,

$$\text{Bott}(\varphi, uv)|_{\mathcal{P}} = \text{Bott}(\varphi, uv_1)|_{\mathcal{P}} + \text{Bott}(\varphi, U^* v_2 U)|_{\mathcal{P}} \quad (\text{e 13.217})$$

$$= 0 + \text{Bott}(\varphi(e_A \cdot e_A), 1) + \text{Bott}(\varphi', v_2)|_{\mathcal{P}} = 0. \quad (\text{e 13.218})$$

Put  $\Psi = \text{Ad } U \circ \psi$ ,  $\psi'' = \text{Ad } U \circ \psi'$  and  $u_2 = U^* v_2 U$ . Put  $\varepsilon_s = s^2 \varepsilon_1$ . We have (recall  $w'_s = w' \otimes 1_s$ )

$$((1_s - \varphi(p_i)) + ((1_s - \varphi(p_i) + (uv)_s \varphi(p_i))) \quad (\text{e 13.219})$$

$$= ((1_s - \varphi(p_i)) + ((1_s - \varphi(p_i) + (uv_1 u_2)_s \varphi(p_i))) \quad (\text{e 13.220})$$

$$\approx_{\varepsilon_s} (1_s - \varphi(p_i) + (uv_1)_s (u_2)_s \Psi(p_i)) \quad (\text{e 13.221})$$

$$\approx_{s^2 K \delta_2} (1_s - \varphi(p_i)) + (uv_1)_s \Psi(p_i) (u_2)_s \Psi(p_i) \quad (\text{e 13.222})$$

$$\approx_{2\varepsilon_s} ((1_s - \varphi(p_i))(1_s - \Psi(p_i)) + (uv_1)_s \varphi(p_i) \Psi(p_i) (u_2)_s \Psi(p_i)) \quad (\text{e 13.223})$$

$$\approx_{2\varepsilon_s} ((1_s - \varphi(p_i) + (uv_1)_s \varphi(p_i))((1_s - \Psi(p_i)) + (u_2)_s \Psi(p_i))). \quad (\text{e 13.224})$$

Put

$$Z_i = \overline{[(1_s - \Psi(p_i)) + (u_2)_s \Psi(p_i)]((1_s - \Psi(q_i)) + (u_2)_s^* \Psi(q_i))}.$$

Then, since we have assumed that  $\psi'(A) \subset \overline{e_{0,b} B e_{0,b}}$ , one computes, by (e 13.201), that

$$\overline{Z_i} = Z'_i, \quad i = 1, 2, \dots, k. \quad (\text{e 13.225})$$

Then, in  $U(M_s(\tilde{B})/CU(M_s(\tilde{B})))$ , for  $i = 1, 2, \dots, k$ ,

$$\overline{[(1_s - \varphi(p_i) + (uv)_s \varphi(p_i))(1_s - \varphi(q_i) + (uv)_s^* \varphi(q_i))]} \quad (\text{e 13.226})$$

$$\approx_{12\varepsilon_s} \overline{[(1_s - \varphi(p_i) + (uv_1)_s \varphi(p_i)) Z_i ((1_s - \varphi(q_i) + (uv_1)_s^* \varphi(q_i))]} \quad (\text{e 13.227})$$

$$= \overline{[(1_s - \varphi(p_i) + (uv_1)_s \varphi(p_i)) ((1_s - \varphi(q_i) + (uv_1)_s^* \varphi(q_i))]} \overline{Z_i} \quad (\text{e 13.228})$$

$$\approx \varphi^\dagger([p_i] - [q_i]) \overline{Z_i} \approx_{\eta/16} \bar{1}. \quad (\text{see(e 13.209)}) \quad (\text{e 13.229})$$

Now back to  $\psi'$ . Let  $\varphi_{00} : A \rightarrow B_W := B \otimes W \otimes e_q$  be defined by  $\varphi_{00} = \psi_{b,w,1/K} \circ \varphi$ . Then

$$\|[\varphi_{00}(g), ((v_2'')^*)^K]\| < K\delta_0 < \delta_1/2 \text{ for all } g \in \mathcal{G}_0. \quad (\text{e 13.230})$$

By the choice of  $\delta_W$  and  $\mathcal{G}_W$  and by applying 13.8, there exists a continuous path of unitaries  $\{V(t) : t \in [0, 1]\}$  in  $B \otimes \widetilde{W} \otimes e_q$  such that  $V(0) = 1_{\tilde{B}_W}$ ,  $V(1) = (v_2'')^K$  and

$$\|[\varphi_{00}(g), V(t)]\| < \min\{\varepsilon_0/4, \delta'_0/2\} \text{ for all } g \in \mathcal{F} \cup \mathcal{G}'_0. \quad (\text{e 13.231})$$

Write  $V(t) = \lambda(t) \cdot 1_{\tilde{B}_W} + \alpha(V(t))$  for some  $\lambda(t) \in \mathbb{T}$  and  $\alpha(V(t)) \in B_W$ . Put

$$v(t) = v_1 U^*(\lambda(t) \cdot \tilde{B} + \alpha(V(t)))U \text{ for all } t \in [0, 1]. \quad (\text{e 13.232})$$

Then we have

$$\|[\varphi(f), v(t)]\| < \min\{\varepsilon_0, \delta''_0\} \text{ for all } f \in \mathcal{F}_0. \quad (\text{e 13.233})$$

Note that  $v(0) = v_1$  and  $v(1) = v$ . □

**Corollary 13.10.** *Let  $A \in \mathcal{B}_T$  have continuous scale. For any  $1 > \varepsilon_0 > 0$  and any finite subset  $\mathcal{F}_0 \subset A$ , there exist  $\delta > 0$  and a finite subset  $\mathcal{G} \subset A$  satisfying the following:*

*For any  $\varepsilon > 0$  and any finite subset  $\mathcal{F} \subset A$  and any homomorphism  $\varphi : A \rightarrow B \otimes Q$  which maps strictly positive elements to strictly positive elements, where  $B \cong B \otimes \mathcal{Z}_0 \in \mathcal{D}_0$  has continuous scale. If  $u \in U(\widetilde{B \otimes Q})$  is a unitary such that*

$$\|[\varphi(x), u]\| < \delta \text{ for all } x \in \mathcal{G}, \quad (\text{e 13.234})$$

*there exists a unitary  $v \in \widetilde{B \otimes U}$  such that*

$$\|[\varphi(f), v]\| < \varepsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 13.235})$$

*a continuous path of unitaries  $\{u(t) : t \in [0, 1]\} \subset U_0(\widetilde{B \otimes Q})$  such that*

$$u(0) = uv, \quad u(1) = 1 \quad (\text{e 13.236})$$

$$\|[\varphi(a), u(t)]\| < \varepsilon_0 \text{ for all } a \in \mathcal{F}_0 \text{ and for all } t \in [0, 1]. \quad (\text{e 13.237})$$

*Proof.* This is a combination of 13.9 and 13.5. Let  $\varepsilon_0 > 0$  and  $\mathcal{F}_0$  be given. Let  $\delta_1 > 0$ ,  $\sigma > 0$ ,  $\mathcal{G}_1 \subset A$  be a finite subset, let  $\{p_1, p_2, \dots, p_k, q_1, q_2, \dots, q_k\}$  be projections of  $M_N(\tilde{A})$  (for some integer  $N \geq 1$ ) such that  $\{[p_1] - [q_1], [p_2] - [q_2], \dots, [p_k] - [q_k]\}$  generates a free subgroup  $G_u$  of  $K_0(A)$ , and  $\mathcal{P} \subset \underline{K}(A)$  be finite subset required by 13.5.

Let  $\delta_0 > 0$  and  $\mathcal{G}_0$  be required by 13.9 for  $\min\{\delta_1, \varepsilon_0\}$  (in place of  $\varepsilon_0$ ),  $\sigma$  and  $\mathcal{G}_1 \cup \mathcal{F}_0$  (in place of  $\mathcal{F}_0$ ) and  $\mathcal{P}$  and  $G_u$ .

Now suppose that  $\varphi$  and  $u$  satisfy the assumption for this pair of  $\delta_0$  and  $\mathcal{G}_0$ . Let  $\varepsilon > 0$  and  $\mathcal{F} \subset A$  be given. Then, by applying 13.9, there is a unitary  $v \in \tilde{B}_1 = B \otimes Q$  and a continuous path of unitaries  $\{v(t) : t \in [0, 1/2]\} \subset \tilde{B}_1$  such that  $v(0) = v$ ,

$$\|[\varphi(f), v]\| < \varepsilon \text{ for all } f \in \mathcal{F}, \quad (\text{e 13.238})$$

$$\|[\varphi(g), v(t)]\| < \varepsilon_0 \text{ for all } g \in \mathcal{F}_0 \quad (\text{e 13.239})$$

$$\text{Bott}(\varphi, uv(1/2))|_{\mathcal{P}} = \{0\}, \quad [uv(1/2)] = 0 \text{ and} \quad (\text{e 13.240})$$

$$\text{dist}(\overline{((1_s - \varphi(p_i)) + (uv(1))_s \varphi(p_i))(1_s - \varphi(q_i)) + (uv(1))_s^* \varphi(q_i)}, \bar{1}) < \sigma, \quad (\text{e 13.241})$$



where  $1_s = 1_{M_s}$  and  $(uv(1))_s = uv(1) \otimes 1_{M_s}$ . Note, since  $B$  is non-unital, it is easy to see that one can always assume  $uv(1/2) \in CU(\tilde{B})$  (see the first few lines of the proof of 20.11 of [17]). It follows from 13.5 that there is a continuous path of unitaries  $\{u(t) : t \in [1/2, 1]\} \subset \tilde{B}_1$  such that  $u(1/2) = uv(1/2)$ ,  $u(1) = 1_{\tilde{B}_1}$  and

$$\|[\varphi(f), u(t)]\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}_0 \text{ for all } t \in [1/2, 1]. \quad (\text{e 13.242})$$

Finally, define  $u(t) = uv(t)$  for  $t \in [0, 1/2]$ .  $\square$

## 14 Finite nuclear dimension

The following follows from the definition immediately.

**Proposition 14.1.** *Let  $A$  be a non-unital separable amenable simple. Then  $A$  has tracially approximate divisible property in the sense of 14.1 of [17] if and only if the following holds:*

*For any  $\varepsilon > 0$ , any finite subset  $\mathcal{F} \subset A$ , any integer  $n \geq 1$  and any non-zero elements  $a_0 \in A_+ \setminus \{0\}$ , there are mutually orthogonal positive elements  $e_i$ ,  $i = 0, 1, 2, \dots, n$ , elements  $w_i$ ,  $i = 1, 2, \dots, n$ , such that  $w_i^* w_i = e_1^2$ ,  $w_i w_i^* = e_i^2$ ,  $i = 1, 2, \dots, n$ ,  $e_0 \lesssim a_0$  and*

$$\|x - \sum_{i=0}^n e_i x e_i\| < \varepsilon \text{ and } \|w_i x - x w_i\| < \varepsilon, \quad 1 \leq i \leq n, \text{ for all } x \in \mathcal{F}. \quad (\text{e 14.1})$$

**Theorem 14.2.** *Let  $A$  be a non-unital separable simple  $C^*$ -algebra with continuous scale and with finite nuclear dimension which satisfies the UCT. Suppose that every tracial state of  $A$  is a  $W$ -trace. Then  $A \in \mathcal{D}_0$ .*

*Proof.* Suppose that  $A$  is tracially approximately divisible. Then, since we assume that every tracial state of  $A$  is a  $W$  trace, by 17.6 of [17] and the proof of 18.6 of [17],  $A \in \mathcal{D}_0$ . In particular,  $A \otimes \mathcal{Z}_0$  and  $A \otimes U$  are in  $\mathcal{D}_0$  for every UHF-algebra of infinite type. Therefore it suffices to show that  $A$  is tracially approximately divisible.

It follows from [51] that  $A \cong A \otimes \mathcal{Z}$ . Put  $B = A \otimes \mathcal{Z}_0$ ,  $B_q = B \otimes Q$  and  $A_q = A \otimes Q$ . Pick a pair of relatively prime supernatural numbers  $\mathfrak{p}$  and  $\mathfrak{q}$ . Let

$$\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}} = \{f \in C([0, 1], Q) : f(0) \in M_{\mathfrak{p}} \text{ and } f(1) \in M_{\mathfrak{q}}\} \text{ and} \quad (\text{e 14.2})$$

$$D \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}} = \{f \in C([0, 1], D \otimes Q) : f(0) \in D \otimes M_{\mathfrak{p}} \text{ and } f(1) \in D \otimes M_{\mathfrak{q}}\} \quad (\text{e 14.3})$$

for any  $C^*$ -algebra  $D$ . Note, by [48],  $\mathcal{Z}$  is a stationary inductive limit of  $\mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$  with trace clapsing connecting map.

Let  $\varepsilon > 0$  and let  $\mathcal{F} \subset A \otimes \mathcal{Z}$  and let  $a_0 \in (A \otimes \mathcal{Z})_+ \setminus \{0\}$ . Put  $\eta = \inf\{d_\tau(a_0) : \tau \in T(A \otimes \mathcal{Z})\}$ . Since  $A$  is assumed to have continuous scale, one may find a positive element  $f_e \in A \otimes \mathcal{Z}$  with  $\|f_e\| = 1$  such that

$$\tau(f_e) > 1 - \eta/16(n+1)^3 \text{ for all } \tau \in T(A \otimes \mathcal{Z}). \quad (\text{e 14.4})$$

We assume that  $f_e \in \mathcal{F}$ . Without loss of generality, we may also assume that  $\mathcal{F} \subset A \otimes \mathcal{Z}_{\mathfrak{p}, \mathfrak{q}}$ . We may further assume, without loss of generality, that there is  $0 < 1/2 < d_0 < 1$  such that  $f(t) = f(1)$  for all  $t > d_0$  and for all  $f \in \mathcal{F}$ . Note  $\text{Aff}(T(B)) = \text{Aff}(T(A))$  and  $U(\tilde{B})/CU(\tilde{B}) = U(\tilde{A})/CU(\tilde{A})$ . There is a  $KK$ -equivalence  $\kappa \in KL(B, A)$  which is compatible to the identifications above which will be denoted by  $(\kappa, \kappa_T, \kappa_{cu})$ . Let  $\varphi_{\mathfrak{p}} : B \otimes M_{\mathfrak{p}} \rightarrow A \otimes M_{\mathfrak{p}}$  and  $\varphi_{\mathfrak{q}} : B \otimes M_{\mathfrak{q}} \rightarrow A \otimes M_{\mathfrak{q}}$  be isomorphisms given by 12.1 and induced by  $(\kappa \otimes [\text{id}_{M_{\mathfrak{p}}}], \kappa_T, \kappa_{cu} \otimes (\text{id}_{M_{\mathfrak{p}}})_{cu})$  and by



$(\kappa \otimes [\text{id}_{M_q}], \kappa_T, \kappa_{cu} \otimes (\text{id}_{M_q})_{cu})$ . Let  $\psi_p : B \otimes M_p \otimes M_q = B \otimes Q \rightarrow A \otimes M_p \otimes M_q = A \otimes Q$  given by  $\psi_p = \varphi_p \otimes \text{id}_{M_q}$  and let  $\psi_q = \varphi_q \otimes \text{id}_{M_p} : B \otimes Q \rightarrow A \otimes Q$ . Then

$$([\psi_q], (\psi_q)_T, \psi_p^\dagger) = ([\psi_p], (\psi_p)_T, \psi_q^\dagger). \quad (\text{e 14.5})$$

Let  $\mathcal{F}_1 = \{f(1) : f \in \mathcal{F}\}$  in  $A \otimes M_p \otimes M_q$ . Let  $\mathcal{G}_{1,b} = \{\psi_q^{-1}(f) : f \in \mathcal{F}_1\}$  and let  $n \geq 1$  be an integer. Fix an  $\varepsilon > 0$ . Put  $C_{00} = C_0((0, 1]) \oplus M_n(C_0((0, 1]))$  and  $C_g = \{(f, 0), (0, f \otimes e_{i,i}), (0, f \otimes e_{1,i}) : 1 \leq i \leq n\}$  form a set of generators:  $f \in C_0((0, 1])$  is the identity function on  $[0, 1]$  and  $\{e_{i,j}\}_{1 \leq i,j \leq n}$  is a system of matrix units for  $M_n$ .  $C_{00}$  is semi-projective. Let  $\delta_c > 0$  satisfying the following: if  $L : C_{00} \rightarrow C'$  is a  $C_g$ - $\delta_c$ -multiplicative completely positive contractive linear map for a  $C^*$ -algebra  $C'$ , there exists a homomorphism  $h_c : C_{00} \rightarrow C'$  such that

$$\|h_c(g) - L(g)\| < \min\{\varepsilon, \eta\}/64(n+1)^3 \text{ for all } g \in C_g. \quad (\text{e 14.6})$$

Let  $\varepsilon_0 = \min\{\varepsilon/(n+1)^3 16, \delta_c/4, \eta/(n+1)^3 16\}$ .

Let  $\delta > 0$  and  $\mathcal{G} \subset A \otimes Q$  be a finite subset required by 13.10 for  $\varepsilon_0$  and  $\mathcal{F}_1$ . Without loss of generality, we may assume that  $\mathcal{G} \subset (A \otimes Q)^1$  and  $\mathcal{F}_1 \subset \mathcal{G}$ . Let  $\varepsilon_1 = \min\{\varepsilon_0/2, \delta/4\}$  and  $\mathcal{G}_1 = \psi_q^{-1}(\mathcal{G}) \cup \mathcal{G}_{1,b}$ .

It follows from 5.3 (see 5.7) that there exists a unitary  $u \in \widetilde{A \otimes Q}$  such that

$$\|u^* \psi_p(g) u - \psi_q(g)\| < \varepsilon_1/4 \text{ for all } g \in \mathcal{G}_1. \quad (\text{e 14.7})$$

Write  $u = \lambda + \alpha(u)$  for some  $\alpha(u) \in A \otimes Q$ . Choose  $e_{00}, e_{01} \in (A \otimes Q)_+$  with  $\|e_{00}\| = \|e_{01}\| = 1$  such that  $e_{00}e_{01} = e_{00}$  and  $\|e_{00}x - x\| < \varepsilon_1/16$  and  $\|x - xe_{00}\| < \varepsilon_1/16$  for all  $x \in \mathcal{G}_1$  and  $x = \alpha(u)$ . We also assume that there is a non-zero  $e'_{00} \in A \otimes Q$   $e'_{00}e_{01} = 0$ . There is a unitary  $u_1 \in \mathbb{C} \cdot 1_{\widetilde{A \otimes Q}} + \overline{e_{00}(A \otimes Q)e_{00}}$  such that  $\|u_1 - u\| < \varepsilon/8$ . Since  $A \otimes Q$  has stable rank one, there is a unitary  $u_2 \in \mathbb{C} \cdot 1_{\widetilde{A \otimes Q}} + \overline{e'_{00}(A \otimes Q)e'_{00}}$  such that  $[u_2] = -[u]$  in  $K_1(A)$ . Put  $u_3 = uu_2$ . Then, since  $e'_{00}e_{01} = 0$ , by (e 14.7),

$$\|u_3^* \psi_p(g) u_3 - \psi_q(g)\| < \varepsilon_1/2 \text{ for all } g \in \mathcal{G}_1. \quad (\text{e 14.8})$$

But now  $u_3 \in U_0(\widetilde{A \otimes Q})$ . There is a continuous path of unitaries  $\{u(t) : t \in [0, d]\} \subset U(\widetilde{A \otimes Q})$  such that  $u(0) = 1$  and  $u(t) = u_3$  for all  $t \in [d, 1]$  and for some  $0 < d_0 < d < 1$ . Define

$$\gamma(f(t)) = \begin{cases} \psi_p^{-1}(u(t)f(t))u(t)^* & t \in [0, d]; \\ \frac{(1-t)}{1-d} \psi_p^{-1}(u(d)f(d)u(d)^*) + \frac{(t-d)}{1-d} \psi_q^{-1}(f(1)) & t \in (d, 1]. \end{cases} \quad (\text{e 14.9})$$

Note that  $\gamma(f) \in B \otimes \mathcal{Z}_{p,q}$ . For  $f \in \mathcal{F}$ , let  $g = \psi_q^{-1}(f(1)) = \psi_q^{-1}(f(d))$ , by (e 14.8),

$$\|g - \psi_p^{-1}(u(d)\psi_q(g)u(d)^*)\| < \varepsilon_1/2. \quad (\text{e 14.10})$$

In other words, if  $f \in \mathcal{F}$ ,

$$\|\psi_p^{-1}(u(d)f(d)u(d)^*) - \psi_q^{-1}(f(1))\| < \varepsilon_1/2 \quad (\text{e 14.11})$$

Let  $\mathcal{F}_2 = \{\gamma(f) : f \in \mathcal{F}\} \subset B \otimes \mathcal{Z}_{p,q}$ . Since  $B \otimes \mathcal{Z}_{p,q} = (B \otimes \mathcal{Z}_0) \otimes \mathcal{Z}_{p,q}$ , there exist mutually orthogonal positive elements  $e_i$ ,  $i = 0, 1, 2, \dots, n$ , elements  $w_i$ ,  $i = 1, 2, \dots, n$ , in  $B \otimes \mathcal{Z}_{p,q}$  such that  $w_i^* w_i = e_1^2$ ,  $w_i w_i^* = e_i^2$ ,  $e_0 e_i = 0$ ,  $i = 1, 2, \dots, n$ , and

$$\|x - \sum_{i=0}^n e_i x e_i\| < \varepsilon_1/4, \quad \|x w_i - w_i x\| < \varepsilon_1/4, \quad 1 \leq i \leq n \text{ for all } x \in \mathcal{F}_2 \text{ and} \quad (\text{e 14.12})$$

$$d_\tau(e_0) \leq \eta/4 \text{ for all } \tau \in T(B \otimes \mathcal{Z}_{p,q}). \quad (\text{e 14.13})$$

Since  $f_e \in \mathcal{F}$ , (e 14.12) also implies that

$$\sum_{i=1}^n \tau(e_i) \geq 1 - \varepsilon_0/4 - \eta/16n^2 \text{ for all } \tau \in T(B \otimes \mathcal{Z}_{\mathfrak{p},\mathfrak{q}}). \quad (\text{e 14.14})$$

Without loss of generality, we may assume that  $e_i(t) = e_i(1)$  and  $w_i(t) = w_i(1)$  for all  $t \in [d_1, 1]$  for some  $d_1 > d > d_0$ .

Let  $\mathcal{G}_2 = \mathcal{G}_1 \cup \{e_i(1), w_i(1) : 1 \leq i \leq n\}$ . By applying 5.3 again, we obtain another unitary  $u_4 \in \widetilde{A \otimes Q}$  such that

$$\|u_4^*(u_3^*\psi_{\mathfrak{p}}(g)u_3)u_4 - \psi_{\mathfrak{q}}(g)\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_2. \quad (\text{e 14.15})$$

Therefore, for any  $g \in \mathcal{G}_1$ ,

$$\|[\text{Ad } u_3 \circ \psi_{\mathfrak{p}}(g), u_4]\| < \varepsilon_1. \quad (\text{e 14.16})$$

It follows from 13.10 that there exists a unitary  $u_5$  and a continuous path of unitaries  $\{v(t) : t \in [d_1, r]\}$  in  $\widetilde{A \otimes Q}$  (for some  $1 > r > d_1$ ) with  $v(r) = u_4u_5$  and  $v(d_1) = 1_{\widetilde{A \otimes Q}}$  such that

$$\|[\text{Ad } u_3 \circ \psi_{\mathfrak{p}}(g), u_5]\| < \varepsilon_1/16 \text{ for all } g \in \mathcal{G}_2 \text{ and} \quad (\text{e 14.17})$$

$$\|[\text{Ad } u_3 \circ \psi_{\mathfrak{p}}(f), v(t)]\| < \varepsilon_0 \text{ for all } f \in \mathcal{F}_1. \quad (\text{e 14.18})$$

It follows from (e 14.15) and (e 14.17) that

$$\|u(r)^*(u_3^*\psi_{\mathfrak{p}}(g)u_3)u(r) - \psi_{\mathfrak{q}}(g)\| < \varepsilon_1/8 \text{ for all } g \in \mathcal{G}_2. \quad (\text{e 14.19})$$

Now define

$$b'_i = \begin{cases} u^*(t)\psi_{\mathfrak{p}}(e_i(t))u(t) & t \in [0, d_1], \\ v^*(t)u_3^*\psi_{\mathfrak{p}}(e_i(t))u_3v(t) & t \in [d_1, r], \ i = 0, 1, 2, \dots, n \text{ and (e 14.20)} \\ \left(\frac{(1-t)}{1-r}v(r)^*u_3^*\psi_{\mathfrak{p}}(e_i(1))u_3v(r)\right) + \frac{(t-r)}{1-r}\psi_{\mathfrak{q}}(e_i) & t \in (r, 1], \end{cases}$$

$$z'_i = \begin{cases} u^*(t)\psi_{\mathfrak{p}}(w_i(t))u(t) & t \in [0, d_1], \\ v^*(t)u_3^*\psi_{\mathfrak{p}}(w_i(t))u_3v(t) & t \in [d_1, r], \ i = 1, 2, \dots, n. \quad (\text{e 14.21)} \\ \left(\frac{(1-t)}{1-r}v(r)^*u_3^*\psi_{\mathfrak{p}}(w_i(1))u_3v(r)\right) + \frac{(t-r)}{1-r}\psi_{\mathfrak{q}}(w_i) & t \in (r, 1], \end{cases}$$

For the next few estimates, recall that  $f(t) = f(1)$  for all  $t \in [d_0, 1]$ ,  $e_i(t) = e_i(1)$  for all  $t \in [d_1, 1]$ , and  $u(t) = u(d)$  for all  $t \in [d, d_1]$ .

For  $t \in [0, d_0]$ , since  $\gamma(f) \in \mathcal{F}_2$ , by (e 14.12),

$$\|f(t) - \sum_{i=0}^n b'_i(t)f(t)b'_i(t)\| < \varepsilon_1 \text{ for all } f \in \mathcal{F}, \quad (\text{e 14.22})$$

For  $t \in [0, d_1]$ , by the definition of  $\gamma(f)$ , by (e 14.11) and (e 14.12), we have

$$f(t) \approx_{\varepsilon_1/2} u(d)^*\psi_{\mathfrak{p}}(\gamma(f(t)))u(d) \approx_{\varepsilon_1/4} \sum_{i=0}^n b'_i(t)u(d)^*\psi_{\mathfrak{p}}(\gamma(f(t)))u(d)b'_i(t) \quad (\text{e 14.23})$$

$$\approx_{\varepsilon_1/2} \sum_{i=0}^n b'_i(t)f(t)b'_i(t) \text{ for all } f \in \mathcal{F}. \quad (\text{e 14.24})$$

For  $t \in [d_1, r]$ , by (e 14.8), (e 14.17), with  $g = \psi_q^{-1}(f(1))$ ,

$$f(t) = f(1) \approx_{\varepsilon_1} u_3^* \psi_p(g) u_3 \approx_{\varepsilon_0} v(t)^* u_3^* \psi_p(g) u_3 v(t) \quad (\text{e 14.25})$$

$$\approx_{\varepsilon_1/2} \text{Ad } u_3 v(t) \circ \psi_p(\gamma(f(t))) \approx_{\varepsilon_1/4} \text{Ad } u_3 v(t) \circ \psi_p\left(\sum_{i=0}^n e_i(t) \gamma(f(t)) e_i(t)\right) \quad (\text{e 14.26})$$

$$\approx_{\varepsilon_1} \sum_{i=0}^n b'_i(t) f(t) b'_i(t). \quad (\text{e 14.27})$$

On  $[r, 1]$ , by the above, and by (e 14.19), as  $e_i(1) \in \mathcal{G}_2$ ,

$$f(t) = f(1) \approx_{3\varepsilon_1} \sum_{i=0}^n b'_i(r) f(r) b'_i(r) \approx_{4(n+1)\varepsilon_1/8} \sum_{i=0}^n b'_i(t) f(t) b'_i(t). \quad (\text{e 14.28})$$

Coming these, we have that

$$\|f - \sum_{i=0}^n b'_i f b'_i\| < (n+1)\varepsilon_1 + \varepsilon_0 < \varepsilon/16(n+1)^2 \text{ for all } f \in \mathcal{F}. \quad (\text{e 14.29})$$

We also compute that

$$\|z'_i f - f z'_i\| < 2\varepsilon_1 + \varepsilon_0, \quad 1 \leq i \leq n, \text{ for all } f \in \mathcal{F}. \quad (\text{e 14.30})$$

By the semi-projectivity of  $C_{00}$  and choice of  $\delta_c$ , we obtain  $b_i, z_j \in A \otimes \mathcal{Z}_{p,q}$ ,  $i = 0, 1, 2, \dots, n$ ,  $j = 1, 2, \dots, n$ , such that

$$\|b_i - b'_i\| < \min\{\varepsilon, \eta\}/(64n^2) \text{ and } \|z_j - z'_j\| < \min\{\varepsilon, \eta\}/(64n^2), \quad (\text{e 14.31})$$

$$b_i b_l = 0 \text{ if } i \neq l, \quad z_i^* z_i = b_1^2, \quad z_i z_i^* = b_i^2, \quad (\text{e 14.32})$$

$i, l = 0, 1, 2, \dots, n$  and  $j = 1, 2, \dots, n$ . By (e 14.29) and (e 14.30),

$$\|f - \sum_{i=0}^n b_i f b_i\| < \varepsilon \text{ and } \|z_i f - f z_i\| < \varepsilon, \quad 1 \leq i \leq n, \text{ for all } x \in \mathcal{F}. \quad (\text{e 14.33})$$

We also estimate, by (e 14.14), that

$$\tau\left(\sum_{i=1}^n b_i\right) > 1 - \eta/2 \text{ for all } \tau \in T(A \otimes \mathcal{Z}). \quad (\text{e 14.34})$$

It follows that  $d_\tau(b_0) < \eta$  for all  $\tau \in T(A \otimes \mathcal{Z})$ . This implies that  $b_0 \lesssim a_0$ . Therefore  $A \otimes \mathcal{Z}$  has the tracial approximate divisible property (see 14.1 ).  $\square$

**Theorem 14.3.** *Let  $A$  and  $B$  be non-unital separable simple  $C^*$ -algebras with finite nuclear dimension which satisfy the UCT. Suppose  $K_0(A)$  and  $K_0(B)$  are torsion. Then  $A \cong B$  if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \quad (\text{e 14.35})$$

Moreover, both  $A$  and  $B$  are stably isomorphic to one of  $B_T$  constructed in section 6.

*Proof.* Let  $e_A \in A_+$  with  $\|e_A\| = 1$  and  $e_B \in B_+$  with  $\|e_B\| = 1$  such that both  $A_0 := \overline{e_A A e_A}$  and  $B := \overline{e_B B e_B}$  have continuous scales. It follows from 20.5 of [17] that every tracial state of  $A$  is a  $W$ -trace. It follows from 14.2 that both  $A_0$  and  $B_0$  are in  $\mathcal{D}_0$ . Then Theorem 12.2 applies.  $\square$

Finally we offer the following result (as Theorem 1.2).

**Theorem 14.4.** *Let  $A$  and  $B$  be two separable simple  $C^*$ -algebras with finite nuclear dimension and which satisfy the UCT. Then  $A \otimes \mathcal{Z}_0 \cong B \otimes \mathcal{Z}_0$  if and only if*

$$(K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \cong (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \quad (\text{e 14.36})$$

(In case that  $\tilde{T}(A) = \emptyset$ , we view  $\Sigma_A = 0$ . Also, there is no order on  $K_0$ -groups.)

*Proof.* First, if  $A$  is infinite, it follows  $\tilde{T}(A) = \emptyset$ . Moreover since  $A$  has finite nuclear dimension, it is purely infinite. Since  $\text{Ell}(A) \cong \text{Ell}(B)$ ,  $\tilde{T}(B) = \emptyset$ . So  $B$  is also not stably infinite. As  $B$  has finite nuclear dimension,  $B$  is also purely infinite. Thus, infinite case is covered by the classification of non-unital purely infinite simple  $C^*$ -algebras (see [24] and [40]).

We now assume both  $A$  and  $B$  are finite. We only need to show the “if” part.

Put  $A_1 = A \otimes \mathcal{Z}_0$  and  $B_1 = B \otimes \mathcal{Z}_0$ . Then we have

$$(K_0(A_1), K_1(A_1), \tilde{T}(A_1), \Sigma_{A_1}) = (K_0(A), K_1(A), \tilde{T}(A), \Sigma_A) \quad (\text{e 14.37})$$

$$(K_0(B_1), K_1(B_1), \tilde{T}(B_1), \Sigma_{B_1}) = (K_0(B), K_1(B), \tilde{T}(B), \Sigma_B). \quad (\text{e 14.38})$$

Let  $e_A \in (A_1)_+$  with  $\|e_A\| = 1$  and  $e_B \in (B_1)_+$  with  $\|e_B\| = 1$  such that  $A_0 := \overline{e_A(A_1)e_A}$  and  $B_0 := \overline{e_B(B_1)e_B}$  have continuous scales. It follows from Proposition 18.5 of [17] that all tracial states of  $A_0 \otimes \mathcal{Z}_0$  and  $B_0 \otimes \mathcal{Z}_0$  are W-traces. It follows from 18.6 of [17] that  $A_0, B_0 \in \mathcal{D}_0$ . It follows from 18.3 of [17] that  $K_0(A_1) = \ker \rho_{A_1}$  and  $K_0(B_1) = \ker \rho_{B_1}$ . Thus the theorem follows from (e 14.37), (e 14.38) and Theorem 12.2.  $\square$

## 15 Appendix

In this appendix, we show that separable amenable  $C^*$ -algebra in  $\mathcal{D}$  are  $\mathcal{Z}$ -stable. The proof is a non-unital version of Matui and Sato’s proof in [36] which is identical to the unital case with only a few modification. We will follow steps of their proof as well as the notation in [36].

**Lemma 15.1** (cf. 2.4 of [36]). *Let  $A$  be a separable simple  $C^*$ -algebra with continuous scale and with  $T(A) \neq \emptyset$  and let  $a \in A_+ \setminus \{0\}$ . Then there exists  $\alpha > 0$  such that*

$$\alpha \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n) \leq \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n^{1/2} a f_n^{1/2}) \quad (\text{e 15.1})$$

for any central sequence  $(f_n)_n$  of positive contractions of  $A$ .

*Proof.* By 9.5 of [17] (see also 7.1 of [17]),  $A$  is strongly uniformly full in  $A$ . Therefore there are  $M(a), N(a) > 0$  such that, for  $b \in A_+$  with  $\|b\| \leq 1$  and for any  $\varepsilon > 0$ , there are  $x_i \in A$  with  $\|x_i\| \leq M(a)$ ,  $i = 1, 2, \dots, N(a)$  such that

$$\left\| \sum_{i=1}^{N(a)} x_i^* a x_i - b \right\| < \varepsilon. \quad (\text{e 15.2})$$

Put  $\alpha_0 = M(a)^2 N(a)$  and  $\alpha = \frac{4}{3\alpha_0}$ . Let  $\{f_n\}_n$  be given. We may assume that

$$\liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n) = \beta > 0.$$

Since  $A$  has continuous scale, there exists  $e \in A_+$  with  $\|e\| = 1$  such that

$$\tau((1 - e^{1/2})c(1 - e^{1/2})) < \beta/8 \text{ for all } \tau \in T(A) \quad (\text{e 15.3})$$

for any  $c \in A_+$  with  $\|c\| = 1$ . Then there are  $x_i \in A$  such that  $\|x_i\| \leq M(a)$ ,  $i = 1, 2, \dots, N(a)$  such that

$$\left\| \sum_{i=1}^{N(a)} x_i^* a x_i - e \right\| < \beta/8, \quad i = 1, 2, \dots \quad (\text{e 15.4})$$

One also has that

$$\tau((1-e)f_n) < \beta/8, \quad n \in \mathbb{N}. \quad (\text{e 15.5})$$

Then, keeping in mind that  $(f_n)_n$  is a central sequence,

$$\begin{aligned} \beta &= \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n) \leq \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(e f) + \beta/8 \\ &\leq \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \sum_{i=1}^{N(a)} \tau(x_i^* a x_i f_n) + \beta/4 \\ &= \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \sum_{i=1}^{N(a)} \tau(x_i^* a^{1/2} f_n a^{1/2} x_i) + \beta/4 \end{aligned} \quad (\text{e 15.6})$$

$$= \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \sum_{i=1}^{N(a)} \tau(f_n^{1/2} a^{1/2} x_i x_i^* a^{1/2} f_n^{1/2}) + \beta/4 \quad (\text{e 15.7})$$

$$\leq \alpha_0 \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n^{1/2} a f_n^{1/2}) + \beta/4. \quad (\text{e 15.8})$$

Thus

$$3\beta/4 \leq \alpha_0 \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n^{1/2} a f_n^{1/2}). \quad (\text{e 15.9})$$

□

**Definition 15.2** (2.1 of [36]). Let  $A$  be a separable  $C^*$ -algebra with  $T(A) \neq \emptyset$  and let  $\varphi : A \rightarrow A$  be a completely positive linear map. Suppose that  $T(A)$  is compact. Recall that  $\varphi$  is said to be excised in small central sequence if for any central sequence  $(e_n)_n$  and  $(f_n)_n$  of positive contractions in  $A$  satisfying

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \tau(e_n) = 0 \quad \text{and} \quad \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n^m) > 0, \quad (\text{e 15.10})$$

there exists  $s_n \in A$  with  $\|s_n\| \leq \|\varphi\|^{1/2}$  and  $n \in \mathbb{N}$  such that

$$\lim_{n \rightarrow \infty} \|s_n^* a s_n - \varphi(a) e_n\| = 0 \quad \text{for all } a \in A \quad \text{and} \quad \lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0. \quad (\text{e 15.11})$$

**Lemma 15.3** (2.5 of [36]). Let  $A$  be a separable simple  $C^*$ -algebra with  $T(A) \neq \emptyset$  with continuous scale. Suppose also that  $A$  has the strict comparison for positive elements. Let  $(e_n)_n$  and  $(f_n)_n$  be as (e 15.10). Then for any  $a \in A_+$  with  $\|a\| = 1$ , there exists a sequence  $(r_n)_n$  in  $A$  such that

$$\lim_{n \rightarrow \infty} \|r_n^* f_n^{1/2} a f_n^{1/2} r_n - e_n\| = 0 \quad \text{and} \quad \limsup_{n \rightarrow \infty} \|r_n\| = \limsup_{n \rightarrow \infty} \|e_n\|^{1/2}. \quad (\text{e 15.12})$$

*Proof.* The proof of this is exactly the same as that of Lemma 2.5 of [36] using 15.1 instead of 2.4 in [36]. □

**Proposition 15.4** (2.2 of [36]). Let  $A$  be a separable amenable simple  $C^*$ -algebra with  $T(A) \neq \emptyset$  and with continuous scale. Suppose that  $A$  has strict comparison for positive elements. Let  $\omega$  be a non-zero pure state of  $A$ ,  $c_i, d_i \in A$ ,  $i = 1, 2, \dots, N$ . Then a completely positive linear map  $\varphi : A \rightarrow A$  defined by  $\varphi(a) = \sum_{i,j=1}^N \omega(d_i^* a d_j) c_i^* c_j$  can be excised by small central sequences.

*Proof.* Let  $\varepsilon > 0$  and let  $\mathcal{F} \subset A$  be a finite subset. It suffices to show that there exists  $s_n \in A$ ,  $n \in \mathbb{N}$ , such that  $\|s_n\| \leq \|\varphi\|^{1/2} + \varepsilon$  and

$$\lim_{n \rightarrow \infty} \|s_n^* a s_n - \varphi(a) e_n\| < \varepsilon \text{ and } \lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0. \quad (\text{e 15.13})$$

Let  $\mathcal{G} = \{d_i^* a d_j : a \in \mathcal{F}, 1 \leq i, j \leq N\}$  and let  $\delta = \varepsilon/N^2$ .

By Proposition 2.2 of [1], there is  $a \in A_+$  with  $\|a\| = 1$  such that  $\|a(\omega(x) - x)a\| < \delta$  for all  $x \in \mathcal{G}$ . Let  $\{e_n\}_n$  and  $\{f_n\}_n$  be as in (e 15.10). By 2.3 of [36], there is a central sequence  $\{\tilde{f}_n\}_n$  of positive contractions of  $A$  such that  $\{\tilde{f}_n f_n\}_n = \{f_n\}_n$  in  $A_\infty$  and

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(\tilde{f}_n^m) = \lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_n^m). \quad (\text{e 15.14})$$

Applying 15.3 to  $\{e_n\}_n$ ,  $\{f_n\}_n$ , and  $a^2$ , we obtain  $r_n \in A$ ,  $n \in \mathbb{N}$ , satisfying

$$\lim_{n \rightarrow \infty} \|r_n \tilde{f}_n^{1/2} a^2 \tilde{f}_n^{1/2} r_n - e_n\| = 0 \text{ and } \limsup_{n \rightarrow \infty} \|r_n\| \leq 1. \quad (\text{e 15.15})$$

Define

$$s_n = \sum_{i=1}^N d_i a \tilde{f}_n^{1/2} r_n c_i, \quad n = 1, 2, \dots \quad (\text{e 15.16})$$

The rest of the proof is exactly the same as that of proof of Proposition 2.2 in [36] with one exception. We need to address the norm of  $s_n$ . Note that, by (e 15.13),

$$\|s_n^* b s_n\| \leq \|\varphi\| + \varepsilon \text{ for all } b \in A_+^1. \quad (\text{e 15.17})$$

Therefore by replacing  $s_n$  by  $E_n s_n$  for some  $E_n \in A_+^1$  as subsequence of an approximate identity of  $A$ , we may assume  $\|s_n\| \leq \|\varphi\|^{1/2}$ .  $\square$

**Lemma 15.5** (3.1 of [36]). *Let  $A$  be a separable amenable simple non-elementary  $C^*$ -algebra, and let  $\omega$  be a non-zero pure state of  $A$ . Then any completely positive contractive linear map  $\varphi : A \rightarrow A$  can be approximated point-wisely in norm by completely positive contractive linear maps  $\psi$  of the form*

$$\psi(a) = \sum_{l=1}^N \sum_{i,j=1}^N \omega(d_i^* a d_j) c_{l,i}^* c_{l,j} \text{ for all } a \in A, \quad (\text{e 15.18})$$

where  $c_{l,i}, d_i \in A$ ,  $l, i = 1, 2, \dots, N$ .

*Proof.* The proof is identical to that of 3.1 of [36]. Unital condition can be easily removed. In the first place that unital condition is mentioned, by using an approximate identity  $\{e_n\}$  of  $A$ , and consider  $\rho(e_n)^{-1/2} \rho(\cdot) \rho(e_n)^{-1/2}$  and  $\sigma(\rho(e_n)^{1/2} \cdot \rho(e_n)^{1/2})$  for some large  $n$ , we can assume that  $\rho(e_n)$  is the unit of  $M_N$ , by considering a hereditary  $C^*$ -subalgebra of a full matrix algebras exactly the way as described in that proof. Then, since we assume that  $A$  is simple and non-elementary,  $\pi(A)$  does not contain any non-zero compact operators on  $\mathcal{H}$  in the second paragraph of that proof. So Voiculescu theorem applies. The rest of proof are unchanged.  $\square$

**Lemma 15.6.** *Let  $A \in \mathcal{D}$  be separable  $C^*$ -algebra with continuous scale. Then, for any integer  $k \geq 1$ , there exists an order zero c.p.c. map  $\psi : M_k \rightarrow A_\infty \cap A'$  such that*

$$\lim_{n \rightarrow \infty} \inf\{|\tau(c_n^m) - 1/k| : \tau \in T(A)\} = 0 \text{ for all } m \in \mathbb{N}, \quad (\text{e 15.19})$$

where  $c_n = \psi(e)$  and  $e \in M_k$  is a minimal rank one projection of  $M_k$ .

*Proof.* This proof can be extracted from the proof of 14.3 of [17]. First keep in mind, by 13.4 of [17],  $A$  has strict comparison for positive elements. In the case that  $A \in \mathcal{D}_0$ , this directly follows from 14.5 of [17]. In this case, by 14.5 of [17], there are two sequences of  $C^*$ -subalgebras  $A_{0,n}$ ,  $M_k(D_n)$  of  $A$ , two sequences of completely positive contractive linear maps  $\varphi_n^{(0)} : A \rightarrow A_{0,n}$  and  $\varphi_n^{(1)} : A \rightarrow D_n \in \mathcal{C}_0^{\theta'}$  satisfy the following:

$$\lim_{n \rightarrow \infty} \|\varphi_n^{(i)}(ab) - \varphi_n^{(i)}(a)\varphi_n^{(i)}(b)\| = 0 \text{ for all } a, b \in A, i = 0, 1, \quad (\text{e 15.20})$$

$$\lim_{n \rightarrow \infty} \|a - \text{diag}(\varphi_n^{(0)}(a), \overbrace{\text{diag}(\varphi_n^{(1)}(a), \varphi_n^{(1)}(a), \dots, \varphi_n^{(1)}(a))}^k)\| = 0 \text{ for all } a \in A, \quad (\text{e 15.21})$$

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} d_\tau(c_n) = 0, \quad (\text{e 15.22})$$

$$\tau(f_{1/4}(\psi_n^{(1)}(a_0))) \geq d \text{ for all } \tau \in T(D_n) \quad (\text{e 15.23})$$

and  $\varphi_n^{(1)}(a_0)$  is a strictly positive element in  $D_n$ , where  $c_n$  is a strictly positive element of  $A_{0,n}$  and  $1 > d > 0$ . It is easy to see (see the proof of 13.1 of [17]) that

$$\lim_{n \rightarrow \infty} \sup\{|\tau(a) - \tau \circ \text{diag}(\overbrace{\varphi_n^{(1)}(a), \varphi_n^{(1)}(a), \dots, \varphi_n^{(1)}(a)}^k)| : \tau \in T(A)\} = 0 \text{ for all } a \in A. \quad (\text{e 15.24})$$

Let  $e_{0,n}$  and  $e_{1,n}$  be approximate identities for  $A_{0,n}$  and  $D_n$ , respectively. Define  $e_{j,l,n} = f_{1/2l}(e_{j,n})$ ,  $j = 0, 1$ ,  $l \in \mathbb{N}$ . Then  $\{e_{0,l,n}\}_l$  and  $\{e_{1,l,n}\}_l$  are approximate identities for  $A_{0,n}$  and

$D_n$ , respectively. Define  $\bar{e}_{1,l,n} = \text{diag}(\overbrace{e_{1,m,n}, e_{1,m,n}, \dots, e_{1,m,n}}^k)$ . Put  $E_{l,n} = e_{0,l,n} + \bar{e}_{1,l,n}$ . Then since  $T(A)$  is compact, as we assume  $A$  has continuous scale,  $\lim_{l \rightarrow \infty} \sup_{\tau \in T(A)} \tau(E_{m,n}) = 1$ .

Therefore, by (e 15.22), it is easy to choose a subsequence  $j_n$  such that

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} |\tau(e_{1,j_n,n}^m) - 1/k| = 0 \text{ for all } m \in \mathbb{N}, \quad (\text{e 15.25})$$

and by (e 15.21),  $\{e_{1,j_n,n}\}$  is a central sequence. Note that we identify  $e_{1,j_n,n}$  with

$\text{diag}(e_{1,j_n,n}, \overbrace{0, \dots, 0}^{k-1}) \subset M_k(D_n)$ . Put  $e_{1,j_n,n,i} = \text{diag}(\overbrace{0, \dots, 0}^{i-1}, e_{1,j_n,n,i}, 0, \dots, 0)$ ,  $i = 1, 2, \dots, k$ . There are  $w_{i,n} \in M_k(D_n)$  such that  $w_{i,n}^* w_{i,n} = e_{1,j_n,n,1}$  and  $w_{i,n} w_{i,n}^* = e_{1,j_n,n,i}$ ,  $i = 2, 3, \dots, k$ . Since  $A$  is stably projectionless, the  $C^*$ -subalgebra generated by  $e_{1,j_n,n,i}$  and  $w_{i,n}$  is isomorphic to  $C_0(C(0, 1], M_k)$ . Note  $\{w_{i,n}\}$  can be chosen to be central (by (e 15.20) and (e 15.21)). Put  $c_n = e_{1,j_n,n}$ . We obtain a completely positive contractive linear map  $\psi : M_k \rightarrow A_\infty \cap A'$ .

In the case that  $A \in \mathcal{D}$ ,  $M_k(D_n)$  is replaced by  $D_n$  and (e 15.21) is replaced by

$$\lim_{n \rightarrow \infty} \|a - \text{diag}(\varphi_n^{(0)}(a), \text{diag}(\varphi_n^{(1)}(a))\| = 0 \text{ for all } a \in A. \quad (\text{e 15.26})$$

But, as in the proof of 14.3 of [17], the algebra  $D$  in that proof is  $\mathcal{Z}$ -stable. Therefore, in the proof of 14.2 of [17], one has that (as (e14,6) there)

$$\|[\varphi_{n,m}(x), y]\| < \varepsilon/16K^2 \text{ for all } x \in \mathcal{F} \quad (\text{e 15.27})$$

and  $y \in \{d''^{1/2}, d'', v'', e_j'', w_j'', j = 1, 2, \dots, K\}$ . Note that one can choose  $K = nk$  and using  $n$  copies of  $e_j''$  and  $w_j''$ , the same argument above also produces the completely positive contractive linear map  $\varphi$  from  $M_k$ .  $\square$

**Lemma 15.7.** *Let  $A$  be a separable amenable simple  $C^*$ -algebra in  $\mathcal{D}$  with continuous scale. Then every completely positive linear map  $\varphi : A \rightarrow A$  can be excised by small central sequences.*



*Proof.* Let  $\varphi : A \rightarrow A$  be a completely positive contractive linear map (so we assume  $\|\varphi\| = 1$  without loss of generality). Let  $\{e_n\}_n$  and  $\{f_n\}_n$  be as in 15.2. By 15.1, we may assume that there exists a pure state  $\omega$  of  $A$  and  $c_{l,i}d_i \in A$ ,  $l, i = 1, 2, \dots, N$ , such that

$$\varphi(a) = \sum_{l=1}^N \sum_{i,j=1}^N \omega(d_i^* a d_j) c_{l,i}^* c_{l,j} \text{ for all } a \in A. \quad (\text{e 15.28})$$

Set  $\varphi_l(a) = \sum_{i,j=1}^N \omega(d_i^* a d_j) c_{l,i}^* c_{l,j}$  for all  $a \in A$ ,  $l = 1, 2, \dots, N$ . Thus  $\varphi = \sum_{l=1}^N \varphi_l$ . Note that Lemma 3.4 of [36] holds for non-unital case, in particular, holds for the case  $A \in \mathcal{D}$  which can also be directly proved by repeatedly using the construction in 15.6 in  $\overline{f_n A f_n}$ . Therefore we also have a central sequence  $\{f_{l,n}\}_n$ ,  $l = 1, 2, \dots, N$ , of positive contractions in  $A$  such that  $\{f_n f_{l,n}\}_n = \{f_{l,n}\}_n$ ,  $\{f_{l,n} f_{l',n}\}_n = 0$ ,  $l \neq l'$ ,  $l = 1, 2, \dots, N$ , in  $A_\infty \cap A'$ , and

$$\lim_{m \rightarrow \infty} \limsup_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(f_{l,n}^m) > 0. \quad (\text{e 15.29})$$

Applying 15.4 to  $\varphi_l$ ,  $\{e_n\}_n$  and  $\{f_{l,n}\}_n$ , we obtain a sequence  $\{s_{l,n}\}_n$  in  $A^1$  such that

$$\lim_{n \rightarrow \infty} \|s_{l,n}^* a s_{l,n} - \varphi_l(a) e_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|f_n s_{l,n} - s_{l,n}\| = 0. \quad (\text{e 15.30})$$

Put  $s_n = \sum_{l=1}^N s_{l,n}$ . One estimates that (recall that  $\|s_{l,n}\| \leq 1$ )

$$\begin{aligned} \|f_n s_n - s_n\| &\leq \sum_{l=1}^N \|f_n s_{l,n} - s_{l,n}\| \\ &\leq \sum_{l=1}^N (\|f_n s_{l,n} - f_n f_{l,n} s_{l,n}\| + \|f_n f_{l,n} s_{l,n} - f_{l,n} s_{l,n}\| + \|f_{l,n} s_{l,n} - s_{l,n}\|) \\ &\leq \sum_{l=1}^N (\|f_n\| \|s_{l,n} - f_{l,n} s_{l,n}\| + \|f_n f_{l,n} - f_{l,n}\| \|s_{l,n}\| + \|f_{l,n} s_{l,n} - s_{l,n}\|) \rightarrow 0, \end{aligned}$$

as  $n \rightarrow \infty$ . If  $l \neq l'$ , then, since  $\{f_{l,n}\}_n$  is central and  $\{f_{l,n} f_{l',n}\}_n = 0$  in  $A_\infty$ ,

$$\lim_{n \rightarrow \infty} \|s_{l,n}^* a s_{l',n}\| = \lim_{n \rightarrow \infty} \|s_{l,n}^* f_{l,n} a f_{l',n} s_{l,n}\| = 0. \quad (\text{e 15.31})$$

Therefore, for all  $a \in A$ ,

$$\lim_{n \rightarrow \infty} \|s_n^* a s_n - \varphi(a) e_n\| = \lim_{n \rightarrow \infty} \left\| \sum_{l=1}^N s_{l,n}^* a s_{l,n} - \varphi_l(a) e_n \right\| = 0. \quad (\text{e 15.32})$$

□

**Definition 15.8** (cf. 4.1 of [36]). Let  $A$  be a separable  $C^*$ -algebra with  $T(A) \neq \emptyset$  and with  $T(A)$  compact. We say  $A$  has property (SI) if for any central sequence  $\{e_n\}_n$  and  $\{f_n\}_n$  which satisfy (e 15.10), there exists a central sequence  $\{s_n\}_n$  in  $A$  such that

$$\lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0 \text{ and } \{s_n^* s_n\}_n - \{e_n\}_n \in A^\perp, \quad (\text{e 15.33})$$

where  $A^\perp = \{\{b_n\}_n \in A_\infty : \{b_n\}_n A = A \{b_n\}_n = 0\}$ .

**Lemma 15.9.** *Let  $A$  be a separable amenable  $C^*$ -algebra in  $\mathcal{D}$  with continuous scale. Then  $A$  has (SI).*

*Proof.* Let  $\{e_n\}_n$  and  $\{f_n\}_n$  be as in (e 15.10). Then, by 15.7,  $\text{id}_A$  can be excised in small central sequences. Thus there is a sequence  $s'_n \in A^1$  such that  $\lim_{n \rightarrow \infty} \|(s'_n)^* a(s'_n) - a e_n\| = 0$  for all  $a \in A$  and  $\lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = 0$ . Fix an approximate identity  $\{d_n\}$  of  $A$ . By passing to  $s'_{n_k}, e'_{n_k}$  and  $f_{n_k}$ , if necessary, we may assume further that

$$\lim_{n \rightarrow \infty} \|(s'_n)^* d_n(s'_n) - d_n e_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|f_n d_n^{1/2} - d_n^{1/2} f_n\| = 0. \quad (\text{e 15.34})$$

Define  $s_n = d_n^{1/2} s'_n$ ,  $n = 1, 2, \dots$ . Then

$$\lim_{n \rightarrow \infty} \|s_n^* s_n - d_n e_n\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|f_n s_n - s_n\| = \lim_{n \rightarrow \infty} \|d_n^{1/2} (f_n s'_n - s'_n)\| = 0. \quad (\text{e 15.35})$$

Moreover, for any  $a \in A$ , since  $\{d_n\}$  is an approximate identity for  $A$ ,

$$\lim_{n \rightarrow \infty} \|a(s_n^* s_n) - a e_n\| \leq \lim_{n \rightarrow \infty} \|a(s'_n)^* d_n(s'_n) - a d_n e_n\| + \lim_{n \rightarrow \infty} \|a d_n e_n - a e_n\| = 0. \quad (\text{e 15.36})$$

It follows that  $\{s_n^* s_n\}_n - \{e_n\}_n \in A^\perp$ . Moreover, for  $a \in A$ , by (e 15.36),

$$\lim_{n \rightarrow \infty} \|[s_n, a]\|^2 = \lim_{n \rightarrow \infty} \|a s_n^* s_n a - a^* s_n^* a s_n - s_n^* a^* s_n a + s_n^* a^* a s_n\| \quad (\text{e 15.37})$$

$$= \lim_{n \rightarrow \infty} \|a s_n^* s_n a - a^* e_n a\| = \lim_{n \rightarrow \infty} \|a(s_n^* s_n - e_n) a\| = 0. \quad (\text{e 15.38})$$

Therefore  $\{s_n\}_n$  is a central sequence.  $\square$

**Theorem 15.10.** *Every separable amenable  $C^*$ -algebra in  $\mathcal{D}$  is  $\mathcal{Z}$ -stable.*

*Proof.* Let  $A \in \mathcal{D}$ . It suffices to show that a non-zero hereditary  $C^*$ -subalgebra of  $A$  is  $\mathcal{Z}$ -stable. Therefore, by 2.3 of [25], we may assume that  $A$  has continuous scale.

Fix any integer  $k > 1$ . By Lemma 15.6, we obtain a central sequence  $\{c_{i,n}\}_n$  in  $A$ ,  $i = 1, 2, \dots, k$ , such that  $\{c_{i,n} c_{j,n}^*\}_n = \delta_{i,j} \{c_{1,n}^2\}_n$  in  $A_\infty$  and

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} |\tau(c_{1,n}^m) - 1/k| = 0 \text{ for all } m \in \mathbb{N}. \quad (\text{e 15.39})$$

Thus we obtain an order zero completely positive contractive linear map  $\varphi : M_k \rightarrow A_\infty \cap A'$  such that  $\varphi(e) = \{c_{1,n}\}_n$  for a minimal projection  $e \in M_k$ . Let  $\{d_n\}$  be an approximate identity for  $A$ . Then  $\{d_n\}_n$  is a central sequence. Then  $\overline{\{d_n\}_n}$  is the identity of  $A_\infty \cap A' / A^\perp$ , where  $\overline{\{d_n\}_n}$  is the image of  $\{d_n\}_n$  in  $A_\infty \cap A' / A^\perp$ . We may choose such  $\{d_n\}$  so that  $\{d_n - \sum_{i=1}^N c_{i,n}^* c_{i,n}\}_n \in (A_\infty)_+$ . Note that, since  $A$  has continuous scale,  $\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \tau(d_n) = 1$ . Let  $\{e_n\}$  be a central sequence of positive contraction such that  $\{e_n\}_n = \{d_n - \sum_{i=1}^k c_{i,n}^* c_{i,n}\}_n$ . As in 15.6  $\{c_{i,n}\}_n$  can be chosen so that

$$\limsup_{n \rightarrow \infty} \sup_{\tau \in T(A)} \tau(e_n) = 0 \quad (\text{e 15.40})$$

which can also be computed directly from (e 15.39). Then, we also have

$$\lim_{m \rightarrow \infty} \liminf_{n \rightarrow \infty} \inf_{\tau \in T(A)} \tau(c_{1,n}^m) = 1/k. \quad (\text{e 15.41})$$

By the property (SI), we obtain a central sequence  $\{s_n\}$  in  $A^1$  such that

$$\{s_n^* s_n\}_n - \{e_n\}_n \in A^\perp \text{ and } \lim_{n \rightarrow \infty} \{c_{1,n} s_n\}_n = \{s_n\}_n \text{ in } A_\infty. \quad (\text{e 15.42})$$

Thus we obtain an order zero completely positive contractive linear map  $\Phi : M_k \rightarrow A_\infty \cap A'/A^\perp$  induced by  $\varphi$  and  $s = \overline{\{s_n\}_n} \in A_\infty \cap A'/A^\perp$  such that,

$$s^*s + \Phi(1_{M_k}) = 1 \text{ and } \Phi(e)s = s \text{ in } A_\infty \cap A'/A^\perp \quad (\text{e 15.43})$$

This implies that  $A \otimes \mathcal{Z} \cong A$  as in the proof of (iv)  $\implies$  (i) in section 4 of [36], see also, for example, Proposition 5.3 and 5.6 of [51]. □

**Remark 15.11.** More general result related to this appendix will appear elsewhere.

## References

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